

Induced subgraphs of graphs with large chromatic number.  
IV. Consecutive holes

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January 17, 2015; revised July 11, 2017

<sup>1</sup>Supported by ONR grant N00014-10-1-0680 and NSF grant DMS-1265563.

### **Abstract**

A *hole* in a graph is an induced subgraph which is a cycle of length at least four. We prove that for all  $\nu > 0$ , every triangle-free graph with sufficiently large chromatic number contains holes of  $\nu$  consecutive lengths. In particular, this implies the first case of a well-known conjecture of Gyárfás [5]: for all  $\ell$ , every triangle-free graph with sufficiently large chromatic number has an odd hole of length at least  $\ell$ .

# 1 Introduction

All graphs in this paper are finite and without loops or parallel edges. A *hole* in a graph is an induced subgraph which is a cycle of length at least four, and a hole is *odd* if its length is odd. A *triangle* in  $G$  is a three-vertex complete subgraph, and a graph is *triangle-free* if it has no triangle. In this paper we are concerned with the chromatic number of triangle-free graphs that have no holes of certain specified lengths.

What can we say about the hole lengths in triangle-free graphs with large chromatic number? There are three well-known conjectures of Gyárfás [5], the third implying the first two, as follows:

**1.1 Conjecture:** *For all  $k, \ell$ , there exists  $n$  such that if  $G$  has no clique of cardinality  $k$  and has chromatic number at least  $n$ , then*

- $G$  has an odd hole;
- $G$  has a hole of length at least  $\ell$ ; and
- $G$  has an odd hole of length at least  $\ell$ .

The first has recently been proved in [6], and the second in [4], but the third remains open. In this paper we prove the third when  $k = 2$ ; that is, we prove:

**1.2** *For all  $\ell$ , there exists  $n$  such that if  $G$  is triangle-free with chromatic number at least  $n$ , then  $G$  has an odd hole of length at least  $\ell$ .*

All that was previously known about the lengths of holes in a triangle-free graph  $G$  with (sufficiently) large chromatic number seems to be:

- $G$  contains an even hole [1] (this is true even if we allow triangles, provided the clique number is bounded);
- $G$  contains an odd hole of length at least seven [3]; and
- $G$  contains a hole of length a multiple of three [2].

Let us say a set  $F$  of integers is *constricting* if there exists  $n$  such that every triangle-free graph with chromatic number at least  $n$  contains a hole with length in  $F$ . Which sets are constricting? Certainly every constricting set is infinite, because there are graphs with arbitrarily large chromatic number and arbitrarily large girth.

Here is basically the only source of counterexamples that we know. Let  $G_1$  be the null graph; for each  $i > 1$ , let  $G_i$  be a triangle-free graph with girth at least  $2^{|V(G_{i-1})|}$  and chromatic number at least  $i$ ; and let  $F$  be the set of all cycle lengths that do not occur in any  $G_i$ . Then  $F$  is not constricting, and yet  $F$  has upper density 1. This shows that not every infinite set is constricting, not even sets with upper density one.

Lower density seems to be closer to the truth. As far as we know, a set is constricting if and only if it has strictly positive lower density, but we are far from proving the implication in either direction. A more approachable question is: suppose that  $F$  contains at least one out of every  $\nu$  consecutive integers; are all such sets  $F$  constricting? We prove a strengthening, the following:

**1.3** For all integers  $\nu > 0$  there exists  $n$  such that if  $G$  is triangle-free with chromatic number at least  $n$ , then for some  $t$ ,  $G$  has a hole of length  $t + i$  for  $1 \leq i \leq \nu$ .

This implies the conjectures of 1.1, and also the result of [2]. We imagine the corresponding result is true for graphs with bounded clique number rather than just triangle-free graphs, but so far we have made no progress in proving this.

## 2 Chromatic number and radius

The proof breaks into three cases, depending on the chromatic number of the subgraphs within a fixed distance of a vertex, so next let us describe that more exactly. If  $X \subseteq V(G)$ , the subgraph of  $G$  induced on  $X$  is denoted by  $G[X]$ , and we often write  $\chi(X)$  for  $\chi(G[X])$ . The *distance* (denoted by  $d_G(u, v)$  or  $d(u, v)$ ) between two vertices  $u, v$  of  $G$  is the length of a shortest path between  $u, v$ , or  $\infty$  if there is no such path. If  $v \in V(G)$  and  $\rho \geq 0$  is an integer,  $N_G^\rho(v)$  or  $N^\rho(v)$  denotes the set of all vertices  $u$  with distance exactly  $\rho$  from  $v$ , and  $N_G^\rho[v]$  or  $N^\rho[v]$  denotes the set of all  $v$  with distance at most  $\rho$  from  $v$ . We denote the maximum over all  $v \in V(G)$  of  $\chi(N_G^\rho[v])$  by  $\chi^\rho(G)$  (setting  $\chi^\rho(G) = 0$  for the null graph).

Since we are only concerned with triangle-free graphs, it follows that  $\chi^1(G) \leq 1$ , but there may be vertices  $v$  such that  $\chi(N_G^2[v])$  is large, and such vertices cause difficulties. If we can find an induced subgraph  $H$  with large chromatic number such that  $\chi^2(H)$  is bounded, then we might as well replace  $G$  by  $H$ . If we cannot find such a subgraph, then we will prove that for all  $\ell \geq 4$ ,  $G$  has a hole of length  $\ell$  (if its chromatic number is large enough in terms of  $\ell$ ).

Next we assume  $\chi^2(G)$  is bounded. If there is an induced subgraph  $H$  with large chromatic number and with  $\chi^3(H)$  bounded, we might as well pass to that; and if not, we prove that  $G$  contains holes of any fixed length (except very short ones) if  $\chi(G)$  is large enough. And the same for  $\chi^\rho(G)$  for all bounded  $\rho$ .

Finally, we assume  $\chi^\rho(G)$  is bounded, for some appropriately large  $\rho$ . (We need  $\rho$  to be exponentially large in terms of  $\nu$ .) In that case we prove that  $G$  contains holes of  $\nu$  consecutive lengths (but the smallest of them might be arbitrarily large).

Let us say this more precisely. Let  $\nu \geq 0$ ; a *hole  $\nu$ -interval* in a graph  $G$  is a sequence  $C_1, \dots, C_\nu$  of holes in  $G$ , such that  $|E(C_{i+1})| = |E(C_i)| + 1$  for  $1 \leq i < \nu$  (thus,  $\nu$  holes with consecutive lengths). Let  $\mathbb{N}$  denote the set of nonnegative integers, and let  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  be a non-decreasing function. For  $\rho \geq 1$ , let us say a graph  $G$  is  $(\rho, \phi)$ -controlled if  $\chi(H) \leq \phi(\chi^\rho(H))$  for every induced subgraph  $H$  of  $G$ . Roughly, this says that in every induced subgraph  $H$  of  $G$  with large chromatic number, there is a vertex  $v$  such that  $H[N_H^\rho[v]]$  has large chromatic number.

We will show the following three statements:

**2.1** Let  $\nu \geq 2$ ; then there exist  $\rho > 0$  and a non-decreasing function  $\phi$  with the following property. If  $G$  is a triangle-free graph then either  $G$  is  $(\rho, \phi)$ -controlled or  $G$  admits a hole  $\nu$ -interval.

**2.2** Let  $\rho > 2$  and  $\ell \geq 4\rho(\rho + 2)$  be integers. For every non-decreasing function  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  there is a non-decreasing function  $\phi'$  with the following property. Let  $G$  be a  $(\rho, \phi)$ -controlled triangle-free graph. Then either  $G$  is  $(2, \phi')$ -controlled or  $G$  has an  $\ell$ -hole.

**2.3** Let  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  be a non-decreasing function; then for all  $\ell \geq 4$  there exists  $n$  such that every  $(2, \phi)$ -controlled triangle-free graph with chromatic number more than  $n$  has an  $\ell$ -hole.

2.3 is easy for  $\ell \leq 6$ , and in another paper [3] (with Maria Chudnovsky) we proved it for  $\ell = 7$ , expecting that to be the easiest of the open cases. By a happy coincidence,  $\ell = 7$  turns out to be the one case that is not handled by the proof method of the present paper.

Let us see that these three together imply 1.3.

**Proof of 1.3, assuming 2.1, 2.2, 2.3.** Let  $\nu \geq 2$ , and let  $\rho$  and  $\phi$  be as in 2.1. Let  $\ell_i = \rho(2\rho + 5)$ , and for  $i = 1, \dots, \nu - 1$  let  $\ell_i = \ell_0 + i$ . By 2.2, for each  $i \in \{0, \dots, \nu - 1\}$  there is a function  $\phi'$  as in 2.2 (with  $\ell$  replaced by  $\ell_i$ ); define  $\phi_i = \phi'$ . Thus  $\phi_0, \dots, \phi_{\nu-1}$  are all non-decreasing functions; define

$$\psi(\kappa) = \max(\phi_0(\kappa), \dots, \phi_{\nu-1}(\kappa))$$

for  $\kappa \geq 0$ . Thus  $\psi$  is non-decreasing. Now by 2.3 (with  $\phi$  replaced by  $\psi$ ) for  $\ell = 4, \dots, \nu - 3$  there exists  $n$  as in 2.3; let  $n_\ell = n$ . Let  $n = \max(n_4, \dots, n_{\nu+3})$ .

We claim that every triangle-free graph with chromatic number more than  $n$  admits a hole  $\nu$ -interval. For let  $G$  be such a graph, and suppose it admits no hole  $\nu$ -interval. From the choice of  $\rho$  and  $\phi$ , it follows that  $G$  is  $(\rho, \phi)$ -controlled. For some  $i \in \{0, \dots, \nu - 1\}$ ,  $G$  has no  $\ell_i$ -hole; so from the choice of  $\phi_i$ ,  $G$  is  $(2, \phi_i)$ -controlled and hence  $(2, \psi)$ -controlled. For some  $\ell \in \{4, \dots, \nu + 3\}$ ,  $G$  has no  $\ell$ -hole; and so from the choice of  $n_\ell$ ,  $\chi(G) \leq n_\ell \leq n$ . This proves 1.3.  $\blacksquare$

The three statements 2.1, 2.2, 2.3 will be proved in separate parts of the paper, and in reverse order. What we are proving is a considerable strengthening of 1.1, and we expect it would be of interest if we show how to prove 1.1 alone. This is much easier, and does not need anything after section 3 of this paper. We sketch a proof of it at the end of section 3.

### 3 Radius 2

In this section we prove 2.3. We begin with the following:

**3.1** *Let  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  be non-decreasing, and let  $G$  be triangle-free and  $(2, \phi)$ -controlled.*

- *If  $\chi(G) > \phi(2)$  then  $G$  has a 5-hole.*
- *If  $\chi(G) > \phi(3)$  then  $G$  has a 6-hole.*
- *If  $\chi(G) > \phi(\phi(\phi(2\phi(2) + 2) + 1) + 1)$  then  $G$  has a 7-hole.*
- *If  $G$  has no 4-hole, then for all  $\ell \geq 5$ , if  $\chi(G) > \phi(2\ell - 9)$  then  $G$  has an  $\ell$ -hole.*

**Proof.** The first statement was proved in [3], but we repeat the proof because it is easy. Suppose that  $\chi(G) > \phi(2)$ , and let  $v$  be a vertex such that  $\chi(G) \leq \phi(\chi(N^2[v]))$ . It follows that  $\chi(N^2[v]) > 2$ , and so there are two adjacent vertices  $x, y \in N^2(v)$ . Since  $G$  is triangle-free,  $x, y, v$ , together with two vertices of  $N^1(v)$  adjacent to  $x, y$  respectively, form a 5-hole.

For the second statement, let  $\chi(G) > \phi(3)$ , and choose a vertex  $v$  such that  $\chi(G) \leq \phi(\chi(N^2[v]))$ . It follows that  $\chi(N^2[v]) > 3$ , and so  $\chi(N^2(v)) > 2$ ; and hence there is an odd hole  $P$  in  $G[N^2(v)]$ . Let  $P$  have vertices  $p_1-p_2-\dots-p_n-p_1$  in order, where  $n \geq 5$ . Choose  $S \subseteq N^1(v)$  minimal such that every vertex in  $V(P)$  has a neighbour in  $S$ . Let  $s_i \in S$  be adjacent to  $p_i$  for  $1 \leq i \leq n$ . ( $s_1, \dots, s_5$  may not all be distinct.) For each  $s \in S$ , some vertex in  $P$  is adjacent to  $s$  and to no other vertex in  $S$ , from the minimality of  $S$ . Consequently we may assume that  $p_3$  is adjacent to  $s_3 \in S$  and has no

other neighbour in  $S$ . If  $p_1$  is nonadjacent to  $s_3$  then  $v-s_1-p_1-p_2-p_3-s_3-v$  is a 6-hole as required, so we may assume that  $p_1$  is adjacent to  $s_3$ , and similarly  $p_5$  is adjacent to  $s_3$ . If  $s_2, s_4$  are nonadjacent to  $p_4, p_2$  respectively then  $v-s_2-p_2-p_3-p_4-s_4-v$  is a 6-hole, so we may assume that one of  $s_2, s_4$  is adjacent both of  $p_2, p_4$ , say  $s_2$ . But then  $s_3-p_1-p_2-s_2-p_4-p_5-s_3$  is a 6-hole.

The third statement is proved in [3].

For the fourth statement, assume that  $G$  has no 4-hole, and let  $\ell \geq 5$  be an integer. Assume that  $\chi(G) > \phi(2\ell - 9)$ , and choose a vertex  $v$  such that  $\chi(G) \leq \phi(\chi(N^2[v]))$ . It follows that  $\chi(N^2(v)) > 2\ell - 9$ . Let  $v_1, \dots, v_t$  be the neighbours of  $v$ , and for  $1 \leq i \leq t$  let  $A_i$  be the set of vertices in  $N^2(v)$  adjacent to  $v_i$ . Since  $G$  has girth at least five, it follows that  $A_1, \dots, A_t$  are disjoint and form a partition of  $N^2(v)$ . Choose  $X \subseteq N^2(v)$  minimal such that  $\chi(X) > 2\ell - 9$ ; and it follows that every vertex of  $G[X]$  has degree at least  $2\ell - 9$  in  $G[X]$ . Choose an integer  $n \geq 1$  maximum such that  $n \leq \ell - 3$  and there exist distinct  $x_1, \dots, x_n \in X$  with the following properties:

- $x_1, \dots, x_n$  are the vertices in order of an induced path of  $G[X]$
- $x_1, \dots, x_n$  all belong to distinct sets  $A_1, \dots, A_t$ .

We may renumber  $v_1, \dots, v_t$  and  $A_1, \dots, A_t$  such that  $x_i \in A_i$  for  $1 \leq i \leq n$ . Certainly  $n \geq 1$ ; suppose that  $n < \ell - 3$ . Now  $x_n$  has at most one neighbour in  $A_i$  for  $1 \leq i \leq n-1$ , since  $G$  has no 4-hole; since  $A_n$  is stable,  $x_n$  has no neighbour in  $A_n$ ; and at most  $n-1$  neighbours of  $x_n$  are adjacent to a vertex in  $\{x_1, \dots, x_{n-1}\}$ . Since  $x_n$  has degree in  $G[X]$  at least  $2\ell - 9 > 2(n-1)$  and  $x_n$  has degree at least  $\ell - 4$  in  $G[X]$ , it follows that  $x_n$  has a neighbour  $y \in X \cap A_j$  for some  $j > n$ , such that  $y$  is nonadjacent to all of  $x_1, \dots, x_{n-1}$ , contrary to the maximality of  $n$ . Thus  $n = \ell - 3$ . Since  $\ell \geq 5$ ,  $v-v_1-x_1-\dots-x_n-v_n-v$  is an  $\ell$ -hole as required. This proves 3.1.  $\blacksquare$

Let  $X \subseteq V(G)$ . A *t-trellis on  $X$  in  $G$*  is a subgraph  $H$  of  $G$  with the following properties.

- $X \subseteq V(H)$ , and  $V(H) \setminus X$  consists of the disjoint union of four sets  $\{a_1, \dots, a_t\}$ ,  $\{b_1, \dots, b_t\}$ ,  $\{a_{x,j} : x \in X, 1 \leq j \leq t\}$  and  $\{b_{x,j} : x \in X, 1 \leq j \leq t\}$ .
- The edges of  $H$  are as follows:
  - $a_j b_j$  for  $1 \leq j \leq t$ ;
  - $x a_{x,j}$  and  $x b_{x,j}$  for  $x \in X$  and  $1 \leq j \leq t$ ; and
  - $a_{x,j} a_j$  and  $b_{x,j} b_j$  for  $x \in X$  and  $1 \leq j \leq t$ .

(Thus, to construct  $H$  we start with  $K_{s,2t}$ , with bipartition  $X$  and  $Y$  say, where  $|X| = s$ ; subdivide all its edges; and then add a matching pairing up the vertices in  $Y$ .)

- For all distinct  $u, v \in V(H)$ , if  $u, v$  are adjacent in  $G$  and nonadjacent in  $H$  then there exist  $x, x' \in X$  and  $j \in \{1, \dots, t\}$  such that  $\{u, v\} = \{a_{x,j}, b_{x',j}\}$ . (In particular,  $X$  is stable.)

We also need a modification of this. An *extended t-trellis on  $X$  in  $G$*  is a subgraph  $H$  of  $G$  with the following properties.

- $X \subseteq V(H)$ , and  $V(H) \setminus X$  consists of the disjoint union of four sets  $\{a_0, a_1, \dots, a_t\}$ ,  $\{b_0, b_1, \dots, b_t\}$ ,  $\{a_{x,j} : x \in X, 0 \leq j \leq t\}$  and  $\{b_{x,j} : x \in X, 0 \leq j \leq t\}$ , together with one more vertex  $c_0$ .

- The edges of  $H$  are as follows:
  - $a_0c_0$  and  $c_0b_0$ ;
  - $a_jb_j$  for  $1 \leq j \leq t$ ;
  - $xa_{x,j}$  and  $xb_{x,j}$  for  $x \in X$  and  $0 \leq j \leq t$ ; and
  - $a_{x,j}a_j$  and  $b_{x,j}b_j$  for  $x \in X$  and  $0 \leq j \leq t$ .
- For all distinct  $u, v \in V(H)$ , if  $u, v$  are adjacent in  $G$  and nonadjacent in  $H$  then there exist  $x, x' \in X$  and  $j \in \{0, \dots, t\}$  such that  $\{u, v\} = \{a_{x,j}, b_{x',j}\}$ .

We need both these definitions; we will show that certain graphs contain extended trellises, and to do so we first show they contain trellises, and then find the extension.

**3.2** *For every integer  $\ell \geq 8$ , there exists  $t \geq 0$  with the following property. Let  $G$  be a graph, let  $X \subseteq V(G)$  with  $|X| = t$ , and let  $H$  be an extended  $t$ -trellis on  $X$ . Then  $G$  has an  $\ell$ -hole.*

**Proof.** By Ramsey's theorem, there exists  $t \geq 0$  such that if  $\mathcal{A}$  is the set of all triples  $(i, i', j)$  with  $1 \leq i < i' \leq t$  and  $1 \leq j \leq t$ , and we partition  $\mathcal{A}$  into two subsets  $\mathcal{A}_1, \mathcal{A}_2$ , then there exist  $R, S \subseteq \{1, \dots, n\}$  with  $|R|, |S| \geq \ell$ , such that the triples  $(i, i', j)$  with  $i < i' \in R$  and  $j \in S$  either all belong to  $\mathcal{A}_1$  or all belong to  $\mathcal{A}_2$ . We claim that  $n$  satisfies the theorem.

For let  $G, X, H$  be as in the theorem. Let  $X = \{x_1, \dots, x_t\}$ , and let  $\mathcal{A}_1$  be the set of all triples  $(i, i', j)$  with  $1 \leq i < i' \leq t$  and  $1 \leq j \leq t$  such that  $a_{i,j}, b_{i',j}$  are nonadjacent, and let  $\mathcal{A}_2$  be the set of all such triples such that  $a_{i,j}, b_{i',j}$  are adjacent. From the choice of  $t$ , we may assume that for some  $k \in \{1, 2\}$ ,  $(i, i', j) \in \mathcal{A}_k$  for all  $i, i', j$  with  $1 \leq i < i' \leq t$  and  $1 \leq j \leq t$ .

For  $1 \leq i < \ell$  let  $P_i$  be the path  $x_i - a_{i,i+1} - a_{i+1,i+1} - b_{i+1,i+1} - x_{i+1}$ . If  $k = 1$  let  $Q_i$  be the path  $x_i - a_{i,i+1} - a_{i+1,i+1} - b_{i+1,i+1} - x_{i+1}$ , and if  $k = 2$  let  $Q_i$  be the path  $x_i - a_{i,i+1} - b_{i+1,i+1} - x_{i+1}$ . Thus  $P_i$  has length four, and  $Q_i$  has length five if  $k = 1$ , and three if  $k = 2$ .

Suppose that  $\ell$  is a multiple of four, say  $\ell = 4p$ . Then the union of  $P_1, \dots, P_{p-1}$  and the path  $x_1 - a_{1,1} - a_{1,p} - x_p$  is a hole of length  $\ell$  as required. Thus we may assume that  $\ell$  is not a multiple of four.

If  $k = 2$ , choose integers  $p, q \geq 0$  such that  $\ell = 4p + 3q$  and  $q > 0$ ; then the union of  $Q_i$  ( $1 \leq i < q$ ),  $P_i$  ( $q \leq i < p + q$ ), and  $x_1 - a_{1,1} - b_{p+q,1} - x_{p+q}$  is the desired hole.

Thus we may assume that  $k = 1$ . If  $\ell \neq 11$ , then, since 4 does not divide  $\ell$ ,  $\ell$  can be expressed as  $4p + 5q$  where  $p, q$  are nonnegative integers and  $q > 0$ ; and the union of  $Q_i$  ( $1 \leq i < q$ ),  $P_i$  ( $q \leq i < p + q$ ), and  $x_1 - a_{1,1} - a_{1,p} - b_{p+q,1} - x_{p+q}$  is the desired hole.

Finally we may assume that  $\ell = 11$ . If  $a_{1,0}, b_{2,0}$  are nonadjacent then the union of  $Q_1$  and  $x_1 - a_{1,0} - a_{0,c_0} - b_{0,b_{2,0}} - x_2$  is the desired hole; while if  $a_{1,0}, b_{2,0}$  are adjacent then the union of  $P_2$ ,  $x_1 - a_{1,0} - b_{2,0} - x_2$ , and  $x_1 - a_{1,3} - a_{3,3} - x_3$  is the desired hole. This proves 3.2.  $\blacksquare$

We remark that we only used the “extended” part of the trellis in 3.2 for the case  $\ell = 11$ . To prove the result just for  $\ell \geq 8$  and  $\ell \neq 11$ , the same proof would work for a (non-extended) trellis.

We also need another definition. Let  $x \in V(G)$ , let  $N$  be some set of neighbours of  $x$ , and let  $C \subseteq V(G)$  be disjoint from  $N \cup \{x\}$ , such that every vertex in  $C$  is nonadjacent to  $x$  and has a neighbour in  $N$ . In this situation we call  $(x, N)$  a *cover* of  $C$  in  $G$ . For  $C, X \subseteq V(G)$ , a *multicover* of  $C$  in  $G$  is a family  $(N_x : x \in X)$  such that

- for each  $x \in X$ ,  $(x, N_x)$  is a cover of  $C$ ;
- for all distinct  $x, x' \in X$ ,  $x'$  has no neighbour in  $\{x\} \cup N_x$  (and in particular all the sets  $\{x\} \cup N_x$  are pairwise disjoint).

If in addition we have

- for all distinct  $x, x' \in X$ , no vertex in  $N_{x'}$  has a neighbour in  $N_x$ ,

we call  $(N_x : x \in X)$  a *stable multicover*.

**3.3** For all  $t, \kappa \geq 0$ , there exist  $\tau, m \geq 0$  with the following property. Let  $G$  be a triangle-free graph such that every induced subgraph of  $G$  with chromatic number more than  $\kappa$  has a 5-hole. Let  $C \subseteq V(G)$  with chromatic number more than  $\tau$ ; and let  $(N_x : x \in X)$  be a multicover of  $C$  with  $|X| \geq m$ . Then there exist  $Y \subseteq X$  with  $|Y| = t$  and an extended  $t$ -trellis on  $Y$  in  $G$ .

**Proof.** For  $0 \leq s \leq t$  let  $m'_s = 5t \cdot 5^{t-s}$ , and let  $m' = m'_0$ . For  $0 \leq s \leq t$  let  $m_s = 5t(20m')^{t-s}$ , and let  $m = m_0$ . Let  $\tau'_t = \kappa + 1$ , and for  $s = t-1, \dots, 0$  let

$$\tau'_s = 5(m'_s + 1) + 5^{m'_s} \tau'_{s+1}.$$

Let  $\tau' = \tau_0$ . Let  $\tau_t = \kappa + 1$ , and for  $s = t-1, \dots, 0$  let

$$\tau_s = 5(m_s + 1) + m_s^{m'+1} 5^{m_s} \tau' + 2^{m_s} 5^{m_s} \tau_{s+1}.$$

Let  $\tau = \tau_0$ . We claim that  $\tau, m$  satisfy the theorem. Let  $G$  be a triangle-free graph such that every induced subgraph of  $G$  with chromatic number more than  $\kappa$  has a 5-hole. We shall prove the following, which implies the theorem:

(1) Let  $C \subseteq V(G)$  and let  $(N_x : x \in X)$  be a multicover of  $C$ , such that either

- $\chi(C) > \tau$  and  $|X| = m$ , or
- $\chi(C) > \tau'$  and  $|X| = m'$  and  $(N_x : x \in X)$  is stable.

Then there exist  $Y \subseteq X$  with  $|Y| = t$  and an extended  $t$ -trellis on  $Y$  in  $G$ .

If  $X' \subseteq X$ , and  $N'_x \subseteq N_x$  for each  $x \in X'$ , and  $C' \subseteq C$ , and every vertex in  $C'$  has a neighbour in  $N'_x$  for each  $x \in X'$ , then  $(N'_x : x \in X')$  is a multicover of  $C'$ , and we say it is *contained in*  $(N_x : x \in X)$ . Consequently, to prove (1), we may assume that:

(2) *Either*

(**Case 1**)  $\chi(C) > \tau$  and  $|X| \geq m$  and there do not exist  $C' \subseteq C$  with  $\chi(C') > \tau'$  and  $X' \subseteq X$  with  $|X'| \geq m'$  and a stable multicover  $(N'_x : x \in X')$  of  $C'$  contained in  $(N_x : x \in X)$ , or

(**Case 2**)  $\chi(C) > \tau'$  and  $|X| \geq m'$  and  $(N_x : x \in X)$  is stable.



Now we construct a  $t$ -trellis on a subset of  $X$  as follows (later we will enlarge it to an extended trellis). We begin with the 0-trellis on  $X$ ,  $H_0$  say, and let  $C_0 = C$ . Inductively, suppose that  $s < t$ , and we have constructed an  $s$ -trellis  $H_s$  on a subset  $X_s \subseteq X$ , with vertex set the disjoint union of  $X_s$ ,  $\{a_1, \dots, a_s\}$ ,  $\{b_1, \dots, b_s\}$ ,  $\{a_{x,j} : x \in X_s, 1 \leq j \leq s\}$  and  $\{b_{x,j} : x \in X_s, 1 \leq j \leq s\}$  in the usual notation, and a subset  $C_s \subseteq C$ , satisfying:

- $a_{x,j}, b_{x,j} \in N_x$  for each  $x \in X_s$  and  $1 \leq j \leq s$ ;
- $a_j, b_j \in C$  for  $1 \leq j \leq s$ ; and
- in case 1,  $|X_s| = m_s$ , and in case 2,  $|X_s| = m'_s$ .
- no vertex in  $V(H_s)$  has a neighbour in  $C_s$ ;
- for each  $v \in C_s$  and each  $x \in X_s$ , there is a neighbour of  $v$  in  $N_x$  that has no neighbour in  $V(H_s)$  except  $x$ ; and
- in case 1,  $\chi(C_s) > \tau_s$ , and in case 2,  $\chi(C_s) > \tau'_s$ .

For each  $x \in X_s$ , let  $N'_x$  be the set of vertices in  $N_x$  with no neighbour in  $V(H_s)$  except  $x$ . Then  $(N'_x : x \in X_s)$  is a multicover of  $C_s$ , and is stable in case 2.

Since  $\chi(C_s) > \tau'_s \geq \kappa$ , there is a 5-hole  $P$  in  $G[C_s]$ , with vertices  $p_1, p_2, \dots, p_5, p_1$  say, in order. For each  $x \in X_s$ , and  $1 \leq i \leq 5$ , let  $D_i(x)$  be the set of vertices in  $N'_x$  adjacent to  $p_i$ , and select  $d_i(x) \in D_i(x)$ . Thus the union of  $V(P)$  and  $\{d_i(x) : 1 \leq i \leq 5, x \in X_s\}$  has cardinality at most  $5(|X_s| + 1)$ , and since  $G$  is triangle-free, there exists  $C_s^1 \subseteq C_s$  with  $\chi(C_s^1) \geq \chi(C_s) - 5(|X_s| + 1)$ , such that no vertex in  $C_s^1$  is adjacent to any of the vertices  $d_i(x)$  or to any vertex in  $P$  (and in particular,  $C_s^1 \cap V(P) = \emptyset$ ).

For each  $x \in X_s$ , no vertex is in more than two of  $D_1(x), \dots, D_5(x)$ , because  $G$  is triangle-free. For each  $v \in C_s^1$  and  $x \in X_s$ , since  $v$  has a neighbour in  $N'_x$ , it follows that there exist adjacent vertices  $p_k, p_{k+1}$  of  $P$  such that some neighbour of  $v$  belongs to  $N'_x \setminus (D_k(x) \cup D_{k+1}(x))$  (reading subscripts modulo 5); choose some such  $k$  and define  $c_x(v) = k$ . There are  $5^{|X_s|}$  possibilities for the  $X_s$ -tuple  $(c_x(v) : x \in X)$ , and so there exists  $C_s^2 \subseteq C_s^1$  with  $\chi(C_s^2) \geq \chi(C_s^1)/5^{|X_s|}$ , such that  $c_x(v) = c_x(v')$  for all  $x \in X_s$  and all  $v, v' \in C_s^2$ . Moreover, since there are only five possibilities for  $c_x(v)$ , there exists  $k \in \{1, \dots, 5\}$  and  $Y_s \subseteq X_s$  with  $|Y_s| = |X_s|/5$  such that  $c_x(v) = k$  for all  $x \in Y_s$  and  $v \in C_s^2$ . Thus  $\chi(C_s^2) \geq (\chi(C_s) - 5(|X_s| + 1))/5^{|X_s|}$ , and so in case 1

$$\chi(C_s^2) > (\tau_s - 5(m_s + 1))/5^{m_s} = m_s^{m'+1} \tau' + 2^{m_s} \tau_{s+1},$$

and in case 2

$$\chi(C_s^2) > (\tau'_s - 5(m'_s + 1))/5^{m'_s} = \tau'_{s+1}.$$

Let  $a_{s+1} = p_k$  and  $b_{s+1} = p_{k+1}$ , and for each  $x \in X_{s+1}$  let  $a_{x,s+1} = d_k(x)$  and  $b_{x,s+1} = d_{k+1}(x)$ .

Next we define  $C_{s+1}$ . In case 2 we define  $C_{s+1} = C_s^2$ ; so henceforth we assume we are in case 1. Let  $Z$  be the union of the sets  $\{a_{x,s+1}, b_{x,s+1}\} (x \in Y_s)$ ; then  $|Z| = 2m_s/5 \leq m_s$ . Let  $z \in Z$ , and let  $Y \subseteq Y_s$  with  $|Y| = m'$ . Let  $D_{z,Y}$  be the set of vertices  $v \in C_s^2$  such that for each  $x \in Y$  there exists a vertex in  $N'_x$  adjacent to both  $v, z$ . For each  $x \in Y$ , let  $N''_x$  denote the set of vertices in  $N'_x$  adjacent to  $z$ ; then  $(N''_x : x \in Y)$  is a multicover of  $D_{z,Y}$ ; and it is stable, since  $G$  is triangle-free. Since we are in case 1, it follows that  $\chi(D_{z,Y}) \leq \tau'$ . Now let  $D_z$  denote the set of vertices  $v \in C_s^2$  such that

for at least  $m'$  values of  $x \in Y_s$  there exists a vertex in  $N'_x$  adjacent to both  $v, z$ ; that is,  $D_z$  is the union of the sets  $D_{z,Y}$  over all choices of  $Y$ . Since there are only at most  $m_s^{m'}$  choices of  $Y$ , it follows that  $\chi(D_z) \leq m_s^{m'} \tau'$ . Thus the union of the sets  $D_z$  over all  $z \in Z$  has chromatic number at most  $m_s^{m'+1} \tau'$ , and so there exists  $C_s^3 \subseteq C_s^2$  with

$$\chi(C_s^3) \geq \chi(C_s^2) - m_s^{m'+1} \tau' > 2^{m_s} \tau_{s+1},$$

such that for every  $v \in C_s^3$ , and every  $z \in Z$ , there are fewer than  $m'$  values of  $x \in Y_s$  such that some vertex in  $N'_x$  is adjacent to both  $v, z$ .

Fix  $v \in C_s^3$  for the moment, and make a digraph  $J_v$  with vertex set  $Y_s$  in which for distinct  $x, y \in Y_s$ ,  $y$  is adjacent from  $x$  in  $J_v$  if some vertex in  $N'_y$  is adjacent to  $v$  and to one of  $a_{x,s+1}, b_{x,s+1}$ . We have just seen that for all  $v$ , every vertex of the digraph  $J_v$  has indegree in  $J$  at most  $2m' - 2$ . It follows that in  $J_v$ , some vertex has indegree plus outdegree at most  $4m' - 4$ , and the same holds for every nonnull subdigraph of  $J_v$ ; and so the undirected graph underlying  $J_v$  can be  $4m'$ -coloured. Hence there is a subset  $U_v$  say of  $Y_s$  of cardinality  $|Y_s|/(4m') = m_{s+1}$  such that no edge of  $J_v$  has both ends in  $U_v$ . There are only  $2^{|Y_s|}$  possibilities for  $U_v$ , and so there exists  $C_s^4 \subseteq C_s^3$  with

$$\chi(C_s^4) \geq \chi(C_s^3)/2^{|Y_s|} > \tau_{s+1}$$

such that the sets  $U_v$  are equal for all  $v \in C_s^4$ . Let  $X_{s+1}$  be this common value of  $U_v$ , and let  $C_{s+1} = C_s^4$ . This completes the definition of  $C_{s+1}$  in case 1.

In both cases, the pairs  $a_j, b_j$  ( $1 \leq j \leq s+1$ ) and the vertices  $a_{x,j}, b_{x,j}$  ( $x \in X_{s+1}, 1 \leq j \leq s+1$ ) define an  $(s+1)$ -trellis  $H_{s+1}$  on  $X_{s+1}$ , and no vertex in  $H_{s+1}$  has a neighbour in  $C_{s+1}$ , and for all  $v \in C_{s+1}$  and  $x \in X_{s+1}$ , some neighbour of  $v$  in  $N_x$  has no neighbour in  $V(H_{s+1})$  except  $x$ . This completes the inductive definition of  $H_s$  and  $C_s$  for  $0 \leq s \leq t$ .

Thus there is a  $t$ -trellis on the set  $X_t$ , where  $|X_t| = 5t$ ; next we need to convert it to an extended  $t$ -trellis on a subset of  $X_t$  of cardinality  $t$ . With the same notation as before (with  $s = t$ ), since  $\chi(C_t) > \tau'_t > \kappa$ , there is a 5-hole  $P$  in  $G[C_t]$ , with vertices  $p_1-p_2-\dots-p_5-p_1$  say, in order. Let  $x \in X_t$ ; a *handle* for  $x$  means a 3-vertex path  $a-c-b$  of  $P$  such that some vertex in  $N'_x$  is adjacent to  $a$ , and not to  $b, c$ , and some vertex in  $N'_x$  is adjacent to  $b$  and not to  $a, c$ . We claim that there is a handle for  $x$ . Choose  $S \subseteq N'_x$  minimal such that every vertex in  $V(P)$  has a neighbour in  $S$ . For  $1 \leq i \leq 5$ , choose  $s_i \in S$  adjacent to  $p_i$ . Suppose first that some  $s_1 \in S$  has only one neighbour in  $V(P)$ , say  $p_1$ . Then no other vertex in  $S$  is adjacent to  $p_1$ , from the minimality of  $S$ , and since  $s_3$  is nonadjacent to  $p_2$  it follows that  $p_1-p_2-p_3$  is a handle for  $x$ . We may assume therefore that each  $s_i$  has at least two (and hence exactly two) neighbours in  $V(P)$ . Let  $s_1$  be adjacent to  $p_1, p_4$  say. From the minimality of  $S$ , one of  $p_1, p_4$  has no more neighbours in  $S$ , say  $p_1$ . But then again  $p_1-p_2-p_3$  is a handle for  $x$ . This proves the claim that for each  $x \in X_t$  there is a handle for  $x$ . Since there are only five possibilities for handles, there exists  $X_0 \subseteq X_t$  with  $|X_0| = |X_t|/5 = t$  such that every vertex in  $X_0$  has the same handle, say  $a_0-c_0-b_0$ . For each  $x \in X_0$  let  $a_{x,0} \in N'_x$  be adjacent to  $a_0$  and not to  $b_0, c_0$ , and let  $b_{x,0}$  be adjacent to  $b_0$  and not to  $a_0, c_0$ . Then the pairs  $a_j, b_j$  ( $1 \leq j \leq t$ ), the path  $a_0-c_0-b_0$ , and the vertices  $a_{x,j}, b_{x,j}$  ( $x \in X_{s+1}, 0 \leq j \leq s+1$ ) define an extended  $t$ -trellis on  $X_0$ . This proves 3.3.  $\blacksquare$

From 3.2 and 3.3 we deduce:

**3.4** *For all  $\kappa \geq 0$  and  $\ell \geq 8$ , there exist  $\tau, m \geq 0$  with the following property. Let  $G$  be a triangle-free graph such that every induced subgraph of  $G$  with chromatic number more than  $\kappa$  has a 5-hole. Let*

$C \subseteq V(G)$  with chromatic number more than  $\tau$ ; and let  $(N_x : x \in X)$  be a multicover of  $C$  with  $|X| \geq m$ . Then  $G$  has an  $\ell$ -hole.

Let  $G$  be a graph and let  $t \geq 0$  be an integer. A  $t$ -cable in  $G$  consists of:

- $t$  distinct vertices  $x_1, \dots, x_t$ , pairwise nonadjacent;
- for  $1 \leq i \leq t$ , a subset  $N_i$  of the set of neighbours of  $x_i$ , such that the sets  $N_1, \dots, N_t$  are pairwise disjoint;
- for  $1 \leq i \leq t$ , disjoint subsets  $Z_{i,i+1}, \dots, Z_{i,t}, Y_i$  of  $N_i$ ; and
- a subset  $C \subseteq V(G)$  disjoint from  $\{x_1, \dots, x_t\} \cup N_1 \cup \dots \cup N_t$

satisfying the following conditions:

- for  $1 \leq i \leq t$ , every vertex in  $C$  has a neighbour in  $Y_i$ , and has no neighbours in  $Z_{i,j}$  for  $i+1 \leq j \leq t$ , and is nonadjacent to  $x_i$ ;
- for  $i < j \leq t$ ,  $x_i$  has no neighbours in  $N_j$ ;
- for  $i < j < k \leq t$ , there are no edges between  $Z_{i,j}$  and  $N_k$ ;
- for all  $i < j \leq t$ , either
  - $Z_{i,j} = \emptyset$  and  $x_j$  has no neighbours in  $Y_i$ , or
  - every vertex in  $N_j$  has a neighbour in  $Z_{i,j}$  and has no neighbours in  $Y_i$ .

We call  $C$  the *base* of the  $t$ -cable, and say  $\chi(C)$  is the *chromatic number* of the  $t$ -cable. Given a  $t$ -cable in this notation, let  $I \subseteq \{1, \dots, t\}$ ; then (after appropriate renumbering) the vertices  $x_i$  ( $i \in I$ ), the sets  $N_i$  ( $i \in I$ ), the sets  $Z_{i,j}$  ( $i, j \in I$ ), the sets  $Y_i$  ( $i \in I$ ) and  $C$  define an  $|I|$ -cable; we call this a *subcable*.

Thus there are two types of pair  $(i, j)$  with  $i < j \leq t$ , and we aim next to apply Ramsey's theorem on these pairs to get a large subcable where all the pairs have the same type. Two special kinds of  $t$ -cables are therefore of interest: *type 1*  $t$ -cables, where for all  $i < j \leq t$ ,  $Z_{i,j} = \emptyset$  and  $x_j$  has no neighbours in  $Y_i$ , and *type 2*  $t$ -cables, where for all  $i < j \leq t$ , every vertex in  $N_j$  has a neighbour in  $Z_{i,j}$  and has no neighbours in  $Y_i$ . A type 1  $t$ -cable with base  $C$  is just a multicover of  $C$  in disguise, so from 3.4 we have:

**3.5** *For all  $\kappa \geq 0$  and  $\ell \geq 8$ , there exist  $\tau, m \geq 0$  with the following property. Let  $G$  be a triangle-free graph such that every induced subgraph of  $G$  with chromatic number more than  $\kappa$  has a 5-hole. If  $G$  admits a type 1  $m$ -cable with chromatic number more than  $\tau$ , then  $G$  has an  $\ell$ -hole.*

We need a similar theorem for type 2 cables.

**3.6** *Let  $G$  be a triangle-free graph. For all  $\ell \geq 5$ , if  $G$  admits a type 2  $(\ell - 3)$ -cable with nonnull base, then  $G$  has an  $\ell$ -hole.*

**Proof.** Let  $t = \ell - 3$  (and so  $t \geq 2$ ) and assume  $G$  contains a type 2  $t$ -cable with nonnull base. In the usual notation, let  $v \in C$ . Since every vertex in  $C$  has a neighbour in  $Y_t$ , there exists  $y_t \in Y_t$  adjacent to  $v$ . Since every vertex in  $N_t$  has a neighbour in  $Z_{t-1,t}$ , there exists  $z_{t-1} \in Z_{t-1,t}$  adjacent to  $y_t$ . Similarly for  $i = t-2, t-3, \dots, 1$  there exists  $z_i \in Z_{i,i+1}$  such that  $z_{i+1}$  is adjacent to  $z_i$ . Thus  $z_1 - z_2 - \dots - z_{t-1} - y_t$  is a path. It is induced; for if  $i, j \leq t$  and  $j \geq i+2$  then  $z_i$  has no neighbour in  $N_j$ , since  $z_i \in Z_{i,i+1}$ . Since  $x_1$  is adjacent to  $z_1$  and to none of  $z_2, \dots, z_{t-1}, y_t$  (because  $t \geq 2$  and  $x_1$  has no neighbours in  $N_j$  for  $j > 1$ ), and  $v$  is adjacent to  $y_t$  and nonadjacent to  $x_1, z_1, \dots, z_{t-1}$ , it follows that

$$x_1 - z_1 - z_2 - \dots - z_{t-1} - y_t - v$$

is an induced path. Now  $v$  has a neighbour  $y_1 \in Y_1$ ; and we claim that  $y_1$  is nonadjacent to  $z_1, \dots, z_{t-1}, y_t$ . Certainly  $y_1, z_1$  are nonadjacent, since they are both adjacent to  $x_1$  and  $G$  is triangle-free. For  $2 \leq j \leq t-1$ ,  $y_1$  is nonadjacent to  $z_j$  since every vertex in  $N_j$  has no neighbours in  $Y_1$ . For the same reason,  $y_1$  is nonadjacent to  $y_t$ , since  $t > 1$ . Consequently

$$x_1 - z_1 - z_2 - \dots - z_{t-1} - y_t - v - y_1 - x_1$$

is a hole of length  $t + 3 = \ell$ . This proves 3.6. ■

We deduce:

**3.7** *For all  $\kappa \geq 0$  and  $\ell \geq 8$ , there exist  $t, \tau \geq 0$  with the following property. Let  $G$  be a triangle-free graph such that every induced subgraph of  $G$  with chromatic number more than  $\kappa$  has a 5-hole. If  $G$  admits a  $t$ -cable with chromatic number more than  $\tau$  then  $G$  has an  $\ell$ -hole.*

**Proof.** Let  $m, \tau$  be as in 3.5. Let  $n = \ell - 3$ . Let  $t$  equal the Ramsey number  $R(m, n)$ ; that is, the smallest integer  $t$  such for every partition of the edges of  $K_t$  into two sets, there is either a  $K_m$  subgraph with all edges in the first set, or a  $K_n$  with all edges in the second. We claim that  $t, \tau$  satisfy the theorem.

For let  $G$  admit a  $t$ -cable with base  $C$  and chromatic number more than  $\tau$ . By Ramsey's theorem either

- there exists  $I \subseteq \{1, \dots, t\}$  with  $|I| = m$  such that for all  $i, j \in I$  with  $i < j$ , every vertex in  $N_j$  has a neighbour in  $Z_{i,j}$  and has no neighbours in  $Y_i$ , or
- there exists  $I \subseteq \{1, \dots, t\}$  with  $|I| = n$  such that for all  $i, j \in I$  with  $i < j$ ,  $Z_{i,j} = \emptyset$  and  $x_j$  has no neighbours in  $Y_i$ .

Thus either there is an  $m$ -subcable of type 1, or an  $n$ -subcable of type 2, with base  $C$  in each case. In the first case the result follows from 3.5, and in the second from 3.6. This proves 3.7. ■

**3.8** *Let  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  be non-decreasing, and let  $t, \tau \geq 0$ . Then there exists  $\tau'$  with the following property. Let  $G$  be a triangle-free graph such that  $G$  is  $(2, \phi)$ -controlled and  $\chi(G) > \tau'$ . Then  $G$  admits a  $t$ -cable with chromatic number more than  $\tau$ .*

**Proof.** Let  $\tau_t = \tau$ , and for  $s = t - 1, \dots, 0$  let  $\tau_s = \phi(2^s \tau_{s+1} + 1)$ ; and let  $\tau' = \tau_0$ . We claim that  $\tau'$  satisfies the theorem. For let  $G$  be a triangle-free graph such that  $G$  is  $(2, \phi)$ -controlled and  $\chi(G) > \tau'$ . Consequently  $G$  admits a 0-cable with chromatic number more than  $\tau_0$ . We claim that for  $s = 1, \dots, t$ ,  $G$  admits an  $s$ -cable with chromatic number more than  $\tau_s$ . For suppose the result holds for some  $s < t$ ; we prove it also holds for  $s+1$ . In the usual notation, since  $\chi(C) > \tau_s = \phi(2^s \tau_{s+1} + 1)$ , there exists  $x_{t+1} \in C$  such that  $\chi(N_{G[C]}^2[x_{t+1}]) > 2^s \tau_{s+1} + 1$ , and hence  $\chi_{G[C]}(N^2(x_{t+1})) > 2^s \tau_{s+1}$ . Let  $D = N_{G[C]}^2(x_{t+1})$ . For each  $v \in D$ , and  $1 \leq i \leq s$ , if some neighbour of  $v$  in  $Y_i$  is nonadjacent to  $x_{s+1}$  define  $c_i(v) = 1$ , and otherwise define  $c_i(v) = 2$ . There are only  $2^s$  possibilities for the  $s$ -tuple  $(c_1(v), \dots, c_s(v))$ , and so there exists  $C' \subseteq D$  with  $\chi(C') \geq \chi(D)/2^s > \tau_{s+1}$  and an  $s$ -tuple  $(c_1, \dots, c_s)$  such that  $c_i(v) = c_i$  for all  $v \in C'$  and  $1 \leq i \leq s$ .

Let  $N_{s+1} = Y'_{s+1}$  be the set of neighbours of  $x_{s+1}$  in  $C$ . For  $1 \leq i \leq s$  define  $Z_{i,s+1}, Y'_i \subseteq Y_i$  as follows:

- if  $c_i = 1$ , let  $Y'_i$  be the set of vertices in  $Y_i$  nonadjacent to  $x_{i+1}$ , and let  $Z_{i,s+1} = \emptyset$
- if  $c_i = 2$ , let  $Y'_i$  be the set of vertices in  $Y_i$  adjacent to  $x_{s+1}$ , and let  $Z_{i,s+1}$  be the set of vertices in  $Y_i$  nonadjacent to  $x_i$ .

Note that in the second case, no vertex in  $Z_{i,s+1}$  has a neighbour in  $C'$ , and no vertex in  $Y'_i$  has a neighbour in  $Y'_{s+1}$ . It follows that  $x_1, \dots, x_{s+1}$ , the sets  $N_1, \dots, N_{s+1}$ , the sets  $Z_{i,j}$  for  $1 \leq i < j \leq s+1$ , the sets  $Y'_i$  for  $1 \leq i \leq s+1$ , and  $C'$ , define an  $(s+1)$ -cable with chromatic number more than  $\tau_{s+1}$ .

This proves that  $G$  admits a  $t$ -cable with chromatic number more than  $\tau_t = \tau$ , and so proves 3.8. ■

Let us put these pieces together to prove 2.3, which we restate:

**3.9** *Let  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  be a non-decreasing function; then for all  $\ell \geq 4$  there exists  $n$  such that every  $(2, \phi)$ -controlled triangle-free graph with chromatic number more than  $n$  has an  $\ell$ -hole.*

**Proof.** If  $\ell \leq 7$  the result follows from 3.1, so we may assume that  $\ell \geq 8$ . Let  $\kappa = \phi(2)$ . By 3.1, every induced subgraph of  $G$  with chromatic number more than  $\kappa$  has a 5-hole. Let  $t, \tau$  be as in 3.7; and let  $\tau'$  be as in 3.8. Let  $n = \tau'$ . We claim that  $n$  satisfies the theorem. For let  $G$  be a  $(2, \phi)$ -controlled triangle-free graph with chromatic number more than  $n$ . By 3.8,  $G$  admits a  $t$ -cable with chromatic number more than  $\tau$ ; and by 3.7,  $G$  has an  $\ell$ -hole. This proves 3.9. ■

If we just want to prove the first conjecture of 1.1, rather than the full strength of 1.3, the remainder of the paper is not needed; let us explain why. The following is proved in [3] (the proof just takes a few lines):

**3.10** *Let  $\ell \geq 3$  and  $\kappa \geq 1$  be integers, and let  $G$  be a graph with no hole of length more than  $\ell$ , such that  $\chi(N(v)), \chi(N^2(v)) \leq \kappa$  for every vertex  $v$ . Then  $\chi(G) \leq (2\ell - 2)\kappa$ .*

For each  $\kappa \geq 0$ , let  $\phi(\kappa) = (2\ell - 2)\kappa$ . It follows from 3.10 that if  $G$  has no hole of length more than  $\ell$ , and  $H$  is an induced subgraph of  $G$  with  $\chi(H) > \phi(\kappa)$ , then  $\chi(N_H^2[v]) > \kappa$  for some vertex  $v$

of  $H$ ; that is,  $G$  is  $(2, \phi)$ -controlled. Then from 3.9 it follows that  $\chi(G)$  is bounded, which proves the first assertion of 1.1. Indeed, we don't even need all of 3.9; instead of an  $\ell$ -hole, we are content with a hole of length at least  $\ell$ , and with this modification 3.9 is easier to prove. For instance, we could get by with trellises instead of extended trellises, since holes of length 11 are of no significance, and indeed we could just use 1-subdivisions of a large  $K_{n,n}$  instead of trellises, since we are not picky about the exact length of the hole.

## 4 Bounded radius

In this section we prove 2.2, which we restate, somewhat reformulated:

**4.1** *Let  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  be non-decreasing, and let  $\rho > 2$  and  $\ell \geq 4\rho(\rho + 2)$  be integers. There is a non-decreasing function  $\phi' : \mathbb{N} \rightarrow \mathbb{N}$ , with the following property. Let  $G$  be a triangle-free graph with no  $\ell$ -hole such that  $G$  is  $(\rho, \phi)$ -controlled. Then  $G$  is  $(2, \phi')$ -controlled.*

4.1 follows immediately from the following.

**4.2** *Let  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  be non-decreasing, and let  $\rho > 2$  and  $\ell \geq 4\rho(\rho + 2)$  be integers. There is a non-decreasing function  $\phi' : \mathbb{N} \rightarrow \mathbb{N}$ , with the following property. Let  $G$  be a triangle-free graph with no  $\ell$ -hole such that  $G$  is  $(\rho, \phi)$ -controlled. Then  $G$  is  $(\rho - 1, \phi')$ -controlled.*

**Proof.** Let  $\ell = 2\alpha\rho + \beta$ , where  $\alpha \geq 0$  is an integer and  $0 \leq \beta < 2\rho$ . Since  $\ell \geq 4\rho(\rho + 2)$ , it follows that  $\alpha \geq 2\rho + 4$ . For  $\kappa \in \mathbb{N}$ , define  $\mu_{\alpha+2}(\kappa) = \phi(0) + 1$ , and for  $h = \alpha + 2, \dots, 2$  define

$$\mu_{h-1}(\kappa) = (\rho + 1)\kappa + \phi(\phi(\mu_h(\kappa) + \kappa) + (2\rho + 2)\kappa),$$

and  $\mu_0(\kappa) = \phi(\mu_1(\kappa) + \kappa)$ . Define  $\phi'(\kappa) = \mu_0(\kappa)$ . We see that  $\phi'$  is non-decreasing.

Let  $G$  be a triangle-free graph with no  $\ell$ -hole such that  $G$  is  $(\rho, \phi)$ -controlled. We will show that  $G$  is  $(\rho - 1, \phi')$ -controlled. Let  $\kappa \in \mathbb{N}$ , such that  $\chi^{\rho-1}(G) \leq \kappa$ ; we must show that  $\chi(G) \leq \mu_0(\kappa)$ . (If so, then the same argument applied to every induced subgraph  $H$  of  $G$  and every  $\kappa$  shows that  $G$  is  $(\rho - 1, \phi')$ -controlled.) Suppose not.

Let  $v \in V(G)$ . Let  $T$  be a path  $v = t_0 t_1 \dots t_\rho$ , such that  $d_G(v, t_\rho) = \rho$ . For the moment fix such a path  $T$ . Let us say a path  $P$  is a  $(v, T)$ -extension if it has the following properties, where  $P$  has vertices  $p_0 p_1 \dots p_n$  in order:

- $P$  is induced, and  $p_0 = t_\rho$ , and  $n \geq \rho$ ;
- $d_G(v, p_i) = \rho$  for  $0 \leq i \leq n$ ;
- $d_G(t_i, p_j) \geq \rho$  for  $0 \leq i \leq \rho$  and  $\rho \leq j \leq n$ ; and
- $d_G(p_i, p_n) \geq \rho$  for  $0 \leq i \leq n - \rho$ .

(1) *If  $P$  as above is a  $(v, T)$ -extension, then  $P \cup T$  is an induced path of length  $\rho + n$ .*

Because  $T$  is induced since  $d_G(v, t_\rho) = \rho$ , and  $P$  is induced by hypothesis. Moreover  $V(P) \cap V(T) = \{t_\rho\}$  since  $d_G(v, t_i) < \rho$  for  $0 \leq i < \rho$ , and  $d_G(v, p_i) = \rho$  for  $0 \leq i \leq n$ . Suppose that some  $t_i$  is

adjacent to some  $p_j$ , where  $i < \rho$  and  $j > 0$ . Since  $d_G(v, p_j) = \rho$  and  $d_G(v, t_i) = i < \rho$ , it follows that  $i = \rho - 1$ . Now  $j \neq 1$  since  $G$  is triangle-free, so  $j \geq 2$ . Since  $d_G(t_{\rho-1}, p_k) \geq \rho$  for  $\rho \leq k \leq n$ , it follows that  $j < \rho$ . Then the path  $t_{\rho-1}-p_j-p_{j+1}-\dots-p_\rho$  has length  $\rho - j + 1 < \rho$ , a contradiction since  $d_G(t_{\rho-1}, p_\rho) \geq \rho$ . This proves (1).

Let  $P, P'$  both be  $(v, T)$ -extensions. We say they are *parallel* if the last three vertices of  $P$  are the same as the last three of  $P'$ , and in particular the last vertices of  $P, P'$  are equal.

(2) Let  $P_1, \dots, P_k$  be  $(v, T)$ -extensions, pairwise parallel. Then there exists  $s \in \{2\rho, 2\rho - 2, 2\rho - 4\}$  such that  $G$  has holes of lengths  $|E(P_1)| + s, \dots, |E(P_k)| + s$ .

Let  $z$  be the common last vertex of  $P_1, \dots, P_k$ , and choose a path  $Z$  between  $v, z$  of length  $\rho$ . Since  $T \cup Z$  is connected, there is an induced path  $Q$  between  $t_\rho, z$  with  $V(Q) \subseteq V(T \cup Z)$ . Let us first examine the length of  $Q$ . Let  $Z$  have vertices  $z_0-z_1-\dots-z_\rho$ , where  $z_0 = v$  and  $z_\rho = z$ . If no vertex in  $\{z_1, \dots, z_\rho\}$  has a neighbour in  $\{t_1, \dots, t_\rho\}$ , then the two sets are disjoint, and  $Q = T \cup Z$  and hence has length  $2\rho$ . We assume then that some  $z_j \in \{z_1, \dots, z_\rho\}$  is adjacent to some  $t_i \in \{t_1, \dots, t_\rho\}$ . Since  $d_G(t_i, z) \geq \rho$  from the definition of a  $(v, T)$ -extension, the path  $t_i-z_j-z_{j+1}-\dots-z_\rho$  has length at least  $\rho$ , and so  $j = 1$ . Since  $z_j$  is adjacent to  $t_0 = v$ , and  $G$  is triangle-free, it follows that  $i \geq 2$ . Since  $d_G(v, t_\rho) = \rho$ , it follows that  $i = 2$ . So there is only one such edge, and in particular the two sets  $\{z_1, \dots, z_\rho\}, \{t_1, \dots, t_{\rho-1}, t_\rho\}$  are disjoint, and  $Q$  has length  $2\rho - 2$ . We have proved then that  $Q$  has length  $2\rho$  or  $2\rho - 2$ .

Now let  $P$  be one of  $P_1, \dots, P_k$ , and let  $P$  have vertices  $p_0-p_1-\dots-p_n$  in order. Thus  $p_0 = t_\rho$  and  $p_n = z_\rho = z$ . Both  $P, Q$  are induced, and their interiors are disjoint, since every vertex  $x$  of the interior of  $Q$  belongs to one of  $V(Z) \setminus \{z\}, V(T) \setminus \{t_\rho\}$  and hence satisfies  $d_G(v, x) < \rho$ , while  $d_G(v, x) = \rho$  for every vertex  $x$  of the interior of  $P$ . Suppose then that some vertex  $x$  in the interior of  $Q$  has a neighbour  $p_j \in \{p_1, \dots, p_{n-1}\}$ . From (1) it follows that  $x \notin V(T)$ , and so  $x \in \{z_1, \dots, z_{\rho-1}\}$ . Since  $d_G(v, p_j) = \rho$ , it follows that  $d_G(v, x) = \rho - 1$ , and so  $x = z_{\rho-1}$ . Consequently  $d_G(p_j, p_n) \leq 2$ , and so  $j > n - \rho$  from the final condition in the definition of a  $(v, T)$ -extension. Since  $d_G(p_{n-\rho}, p_n) \geq \rho$  from the same condition, it follows that the path  $p_{n-\rho}-p_{n-\rho+1}-\dots-p_j-z_{\rho-1}-p_n$  has length at least  $\rho$ , and so  $j \geq n - 2$ . Now  $j \neq n - 1$  since  $G$  is triangle-free, and  $j \neq n$  by its definition, so  $j = n - 2$ .

Consequently there is at most one edge joining the interiors of  $P, Q$ , and any such edge is between  $z_{\rho-1}$  and  $p_{n-2}$ . Let  $s = |E(Q)|$  if there is no such edge, and  $|E(Q)| - 2$  if there is such an edge. In either case  $G$  has a hole of length  $|E(P)| + s$ . Moreover, since the final three vertices of  $P_1, \dots, P_k$  are the same, it follows that  $G$  has a hole of length  $|E(P_i)| + s$  for  $1 \leq i \leq k$ . This proves (2).

Since  $\chi(G) > \mu_0(\kappa)$ , there exists  $z_0$  such that  $\chi(N_G^\rho[z_0]) > \mu_1(\kappa) + \kappa$ , and hence  $\chi(N_G^\rho(z_0)) > \mu_1(\kappa)$ . Let  $H_0 = G$  and let  $T_0$  be the one-vertex subgraph with vertex  $z_0$ . For  $1 \leq h \leq \alpha + 2$ , we define  $y_h, y'_h, S_h, z_h, T_h, M_h, H_h$  as follows. Assume we have defined  $H_{h-1}, T_{h-1}$  and  $z_{h-1}$  such that  $\chi(N_{H_{h-1}}^\rho(z_{h-1})) > \mu_{h-1}(\kappa)$  and  $T_{h-1}$  is an induced path of  $G$  with at most  $\rho + 1$  vertices and with one end  $z_{h-1}$ . Let  $M_h$  be the subgraph induced on the set of all vertices  $v$  of  $H_{h-1}$  that satisfy

- $d_{H_{h-1}}(z_{h-1}, v) = \rho$ ; and
- $d_G(x, v) \geq \rho$  for every vertex  $x$  of  $T_{h-1}$ .

Since  $\chi(N^{\rho-1}(x)) \leq \kappa$  for each vertex  $x$  of  $T_{h-1}$ , and  $\chi(N_{H_{h-1}}^\rho(z_{h-1})) > \mu_{h-1}(\kappa)$ , it follows that

$$\chi(M_h) > \mu_{h-1}(\kappa) - (\rho + 1)\kappa = \phi(\phi(\mu_h(\kappa) + \kappa) + (2\rho + 2)\kappa).$$

Since  $G$  is  $(\rho, \phi)$ -controlled, there is a vertex  $y_h \in M_h$  such that

$$\chi(N_{M_h}^\rho[y_h]) > \phi(\mu_h(\kappa) + \kappa) + (2\rho + 2)\kappa,$$

and hence with

$$\chi(N_{M_h}^\rho(y_h)) > \phi(\mu_h(\kappa) + \kappa) + (2\rho + 1)\kappa.$$

Let  $S_h$  be a path of  $H_{h-1}$  of length  $\rho$  between  $z_{h-1}$  and  $y_h$ . Let  $y'_h$  be adjacent to  $y_h$  in  $M_h$ . Let  $S'_h$  be a path of  $H_{h-1}$  of length  $\rho$  between  $z_{h-1}$  and  $y'_h$ . Let  $H_h$  be the subgraph induced on the set of all vertices  $v$  of  $M_h$  with the following properties:

- $d_{M_h}(y_h, v) = \rho$ ; and
- $d_G(x, v) \geq \rho$  for every  $x \in V(S_h) \cup V(S'_h)$ .

Since  $\chi(N_{M_h}^\rho(y_h)) > \phi(\mu_h(\kappa) + \kappa) + (2\rho + 1)\kappa$ , and  $\chi(N^{\rho-1}(x)) \leq \kappa$  for each vertex  $x$  of  $V(S_h \cup S'_h)$ , and there are at most  $2\rho + 1$  such vertices  $x$ , it follows that  $\chi(H_h) > \phi(\mu_h(\kappa) + \kappa)$ . Consequently there exists  $z_h \in H_h$  such that  $\chi(N_{H_h}^\rho[z_h]) > \mu_h(\kappa) + \kappa$ , and hence with  $\chi(N_{H_h}^\rho(z_h)) > \mu_h(\kappa)$ . Let  $T_h$  be a path of  $M_h$  of length  $\rho$  between  $y_h, z_h$ . This completes the inductive definition of  $y_h, y'_h, S_h, z_h, T_h, M_h, H_h$  for  $1 \leq h \leq \alpha + 2$ .

(3) For  $1 \leq h \leq \alpha + 2$ ,  $S_h \cup T_h$  is an induced path  $L_h$  between  $z_{h-1}, z_h$  of length  $2\rho$ . Also there is an induced path  $L'_h$  between  $z_{h-1}, z_h$  with  $V(L'_h) \subseteq V(S'_h \cup T_h)$  of length  $2\rho - 1$  or  $2\rho + 1$ .

The first claim follows from (1). For the second, the graph formed by the union of  $S'_h, T_h$  and the edge  $y_h y'_h$  is a path, but it might not be induced. If it is induced, it has length  $2\rho + 1$  as required; and since  $S'_h$  and  $T_h$  are both induced paths, we may assume that some vertex  $a$  of  $S'_h$  is adjacent to some vertex  $b$  of  $T_h$ , where  $(a, b) \neq (y'_h, y_h)$ . Since every vertex of  $S'_h$  has distance at most  $\rho - 2$  from  $z_{h-1}$  except the last two, and every vertex of  $T_h$  has distance at least  $\rho$  from  $z_{h-1}$ , it follows that  $a$  is either  $y'_h$  or its neighbour in  $S'_h$ . Now  $d_G(y'_h, z_h) = \rho$ , so  $y'_h$  has no neighbour in  $T_h$  except for  $y_h$  (because  $y'_h$  is not adjacent to the second vertex of  $T_h$  since  $G$  is triangle-free). Thus  $a$  is the penultimate vertex of  $S'_h$ . Consequently  $b \neq y_h$  since  $G$  is triangle-free, and since  $d_G(a, z_h) \geq \rho$ ,  $a$  has no neighbour in  $T_h$  different from the second vertex of  $T_h$ . We deduce that  $b$  is indeed the second vertex of  $T_h$ ; and so there is an induced path between  $z_{h-1}, z_h$  of length  $2\rho - 1$  with vertex set a subset of  $V(S'_h \cup T_h)$ . This proves (3).

Let there be  $q$  values of  $h \in \{4, \dots, \alpha + 2\}$  such that  $L'_h$  has length  $2\rho - 1$ . For  $4 \leq h \leq \alpha + 2$ , choose  $L''_h \in \{L_h, L'_h\}$ ; then  $L''_4 \cup L''_5 \cup \dots \cup L''_{\alpha+2}$  is an induced path between  $z_3$  and  $z_{\alpha+2}$ , and it is a  $(y_3, T_3)$ -extension, for every choice of  $L''_4, L''_5, \dots, L''_{\alpha+2}$ . Moreover, all these  $(y_3, T_3)$ -extensions are parallel (since the last  $\rho$  vertices of  $L_{\alpha+2}, L'_{\alpha+2}$  are the same). These paths have lengths every integer between  $2\rho(\alpha - 1) - q$  and  $(2\rho + 1)(\alpha - 1) - q$ , that is, every integer between  $\ell - \beta - q - 2\rho$  and  $\ell + \alpha - \beta - q - 2\rho - 1$ . From (2),  $G$  has holes of every length between  $\ell - \beta - q$  and  $\ell + \alpha - \beta - q - 5$ . Since  $G$  has no  $l$ -hole, it follows that  $\ell + \alpha - \beta - q - 5 < \ell$ , that is,  $\alpha \leq \beta + q + 4$ . But by concatenating each of the paths  $L''_4 \cup L''_5 \cup \dots \cup L''_{\alpha+2}$  with  $L_3$ , we obtain a  $(y_2, T_2)$ -extension of length exactly  $2\rho$  more; and so there are  $(y_2, T_2)$ -extensions of all lengths between  $\ell - \beta - q$  and  $\ell + \alpha - \beta - q - 1$ . Hence by (2) there are holes in  $G$  of all lengths between  $\ell - \beta - q + 2\rho$  and  $\ell + \alpha - \beta - q + 2\rho - 5$ . Since  $\beta + q \geq \alpha - 4 \geq 2\rho$ , it follows that  $\ell - \beta - q + 2\rho \leq \ell$ . Consequently  $\ell + \alpha - \beta - q + 2\rho - 5 < \ell$ , since



there is no  $\ell$ -hole, that is,  $\alpha + 2\rho \leq \beta + q + 4$ . Similarly, by concatenating all these  $(y_2, T_2)$ -extensions with  $L_2$ , we obtain  $(y_1, T_1)$ -extensions of all lengths between  $\ell - \beta - q + 2\rho$  and  $\ell + \alpha - \beta - q + 2\rho - 1$ . By (2), there are holes of all lengths between  $\ell - \beta - q + 4\rho$  and  $\ell + \alpha - \beta - q + 4\rho - 5$ . But  $\ell - \beta - q + 4\rho \leq \ell$ , since  $\beta + q \geq \alpha + 2\rho - 4 \geq 4\rho$ , and yet

$$\ell + \alpha - \beta - q + 4\rho - 5 = \ell + 2\rho - 3 + (\alpha - 1 - q) + (2\rho - 1 - \beta) \geq \ell$$

since  $q \leq \alpha - 1$  and  $\beta \leq 2\rho - 1$ . Consequently there is an  $\ell$ -hole, a contradiction. This proves 4.2 and hence 4.1.  $\blacksquare$

## 5 Showers

Now we come to the third and most complicated part of the proof: proving 2.1. This will occupy the remainder of the paper.

What can we prove about hole lengths if  $\chi^\rho(G)$  is bounded for some large fixed  $\rho$ ? In 4.1 we were able to guarantee the presence of a hole of any desired length (almost), but in these new circumstances that becomes impossible; for any fixed  $\rho \geq 0$  and  $\ell \geq 2$ , there are graphs with arbitrarily large  $\chi$ , and girth more than  $\max(\ell, \rho/2)$ ; which implies that  $\chi^\rho(G)$  is at most 2, and yet they have no  $\ell$ -hole. We will show the following, a reformulation of 2.1.

**5.1** *Let  $\nu \geq 2$  and  $\kappa \geq 0$  be integers, and let  $G$  be a triangle-free graph such that  $\chi^\rho(G) \leq \kappa$ , where  $\rho = 3^{\nu+2} + 4$ . If  $G$  admits no hole  $\nu$ -interval then  $\chi(G)$  is bounded.*

The proof will need a number of steps and preliminary lemmas. We begin with some definitions. A *levelling* in  $G$  is a sequence of pairwise disjoint subsets  $(L_0, L_1, \dots, L_k)$  of  $V(G)$  such that

- $|L_0| = 1$ ;
- for  $1 \leq i \leq k$  every vertex in  $L_i$  has a neighbour in  $L_{i-1}$ ;
- for  $0 \leq i < j \leq k$ , if  $j > i + 1$  then no vertex in  $L_j$  has a neighbour in  $L_i$ .

We call  $L_k$  the *base* of the levelling. The *chromatic number* of a levelling is the chromatic number of its base. We observe first:

**5.2** *For any integer  $\tau \geq 0$ , if  $\chi(G) > 2\tau$  then  $G$  admits a levelling with chromatic number more than  $\tau$ .*

**Proof.** Choose a component  $C$  of  $G$  with chromatic number equal to that of  $G$ , and let  $z$  be a vertex in that component. For each  $i \geq 0$ , let  $L_i$  be the set of vertices  $v$  of  $C$  such that  $d_C(z, v) = i$ , and choose  $j$  such that  $L_0 \cup \dots \cup L_j = V(C)$ . If  $\chi(L_k) \leq \tau$  for all  $k$  with  $0 \leq k \leq j$ , then  $\chi(C) \leq 2\tau$  (take two disjoint sets of colours both of size  $\tau$ , and use them for the even and odd levels alternately), which is impossible; so there exists  $k$  such that  $\chi(L_k) > \tau$ . Then  $(L_0, \dots, L_k)$  is the desired levelling. This proves 5.2.  $\blacksquare$

If  $(L_0, \dots, L_k)$  is a levelling in  $G$ , we call the unique vertex in  $L_0$  the *head* of the levelling, and we call  $L_0 \cup \dots \cup L_k$  the *vertex set* of the levelling. A path  $P$  of  $G[V]$  (where  $V$  is the vertex set of the levelling) with ends  $u, v$  is *monotone* (with respect to the given levelling) if there exist  $h, j$  with  $0 \leq h, j \leq k$ , such that  $u \in L_h, v \in L_j$ , and  $P$  has length  $|j - h|$ ; and therefore  $P$  has exactly one vertex in  $L_i$  for each  $i$  between  $h, j$ , and has no other vertices.

There is a notational problem with levellings: that while it seems most natural to number levels starting with the head as level zero, most of the action will be at or close to the base  $L_k$ , and we constantly have to refer to the parameter  $k$ . To obviate this, let us say a vertex  $v$  of the vertex set has *height*  $k - i$  if  $v \in L_i$  where  $0 \leq i \leq k$ . Thus vertices in  $L_k$  have height zero.

A *shower* in  $G$  is a sequence  $(L_0, L_1, \dots, L_k, s)$  where  $L_0, L_1, \dots, L_k$  are pairwise disjoint subsets of  $V(G)$  and  $s \in L_k$ , such that

- $|L_0| = 1$ ;
- for  $1 \leq i < k$  every vertex in  $L_i$  has a neighbour in  $L_{i-1}$ ;
- for  $0 \leq i < j \leq k$ , if  $j > i + 1$  then no vertex in  $L_j$  has a neighbour in  $L_i$ ; and
- $G[L_k]$  is connected.

The differences between a shower and a levelling are that, first, not every vertex in  $L_k$  needs to have a neighbour in  $L_{k-1}$  (and indeed, there may be no edges between  $L_{k-1}$  and  $L_k$ , although such showers will not be of interest); second, that  $G[L_k]$  is connected; and third, the distinguished vertex  $s$ . We call  $L_0, \dots, L_k$  the *levels* of the shower, and  $s$  the *drain* of the shower. We define “head”, “base”, “vertex set”, “monotone”, “height” for showers just as for levellings. The set of vertices in  $L_k$  with a neighbour in  $L_{k-1}$  is called the *floor* of the shower.

If  $\mathcal{S} = (L_0, \dots, L_k, s)$  is a shower, with head  $z_0$  and vertex set  $V$ , a *recirculator* for  $\mathcal{S}$  is an induced path  $R$  with ends  $s, z_0$  such that no internal vertex of  $R$  belongs to  $V$  and no internal vertex of  $R$  has any neighbours in  $V \setminus \{s, z_0\}$ . The *distance*  $d_G(P_1, P_2)$  between two nonnull subgraphs  $P_1, P_2$  of  $G$  is the minimum of  $d_G(v_1, v_2)$  over all  $v_1 \in V(P_1)$  and  $v_2 \in V(P_2)$ .

**5.3** Let  $\tau, \kappa \geq 0$  be integers. Let  $G$  be a graph such that  $\chi^8(G) \leq \kappa$ . Let  $(L_0, \dots, L_k)$  be a levelling in  $G$ , where  $\chi(L_k) > 22\tau + 2\kappa$ . Then there is a shower  $(V_0, \dots, V_n, s)$  in  $G$ , with floor of chromatic number more than  $\tau$ , and with a recirculator, such that

- $V_n \subseteq L_k$ , and  $V_{n-1} \subseteq L_{k-1}$ ; and
- $V_0, \dots, V_{n-2} \subseteq L_0 \cup \dots \cup L_{k-2}$ .

**Proof.** By replacing  $L_k$  by the vertex set of a component of  $G[L_k]$  with maximum chromatic number, we may assume that  $G[L_k]$  is connected. A *post* is a monotone path with an end in  $L_k$ . Since  $\chi(L_k) > \kappa$ , there exist two vertices of  $L_k$  with distance more than 8. It follows that there are two posts both of length three with distance at least three. Consequently we can choose two posts  $P, Q$  with the following properties:

- $P, Q$  have the same length  $k - h \geq 3$ ;
- $d_G(P, Q) \geq 3$ ;

- subject to these two conditions,  $h$  is minimum.

Let  $P$  have vertices  $p_k-p_{k-1}-\dots-p_h$  and  $Q$  have vertices  $q_k-q_{k-1}-\dots-q_h$ , where  $p_i, q_i \in L_i$  for  $h \leq i \leq k$ . Let  $p_{h-1}, q_{h-1}$  be parents of  $p_h, q_h$  respectively. From the minimality of  $h$ , either

- $p_{h-1}, q_{h-1}$  are adjacent, or
- some vertex is adjacent to  $p_{h-1}$  and to at least one of  $q_{h-1}, q_h, q_{h+1}$ , or
- some vertex is adjacent to  $q_{h-1}$  and to at least one of  $p_{h-1}, p_h, p_{h+1}$ .

In each case there is a connected induced subgraph  $M$  with  $V(M) \subseteq L_0 \cup \dots \cup L_h \cup \{p_{h+1}, q_{h+1}\}$ , with at most seven vertices, and with  $p_{h+1}, p_h, p_{h-1}, q_{h+1}, q_h, q_{h-1} \in V(M)$ ; and if there is a vertex in  $V(M) \setminus V(P \cup Q)$ , then it belongs to  $L_{h-2} \cup L_{h-1} \cup L_h$ , and has a neighbour in  $\{p_{h+1}, p_h, p_{h-1}\}$  and one in  $\{q_{h+1}, q_h, q_{h-1}\}$ .

Let  $X$  be the set of vertices  $x \in L_{k-1}$  such that there is a path  $R$  from  $x$  to  $p_{h+1}$  satisfying:

- $R$  has length at most  $k - h + 8$ ;
- every internal vertex of  $R$  belongs to  $L_0 \cup \dots \cup L_{k-2}$ ; and
- no vertex of  $R \setminus p_{h+1}$  equals or is adjacent to any vertex in  $\{p_{h+2}, \dots, p_k\}$ .

Define  $Y \subseteq L_{k-1}$  similarly with  $P, Q$  exchanged.

(1) *Every vertex  $v \in L_k$  with  $d_G(v, p_k), d_G(v, q_k) \geq 7$  has a neighbour in  $X \cup Y$ .*

Let  $v \in L_k$  with  $d_G(v, p_k), d_G(v, q_k) \geq 7$ , and let  $r_0-r_1-\dots-r_k = v$  be a path between  $r_0 \in L_0$  and  $v = r_k$ . We claim that  $r_{k-1} \in X \cup Y$ . From the minimality of  $h$ , one of  $r_{h-1}, \dots, r_k$  has distance at most two from one of  $p_{h-1}, \dots, p_k$ . Choose  $j$  maximum such that  $r_j$  has distance at most two from some vertex  $u$  say of  $P \cup Q \cup M$ . Thus  $j \geq h-1$ . If  $j = k$ , then  $u \notin V(M) \setminus V(P \cup Q)$  because  $k-h \geq 3$ , and so  $u$  is one of  $p_k, p_{k-1}, p_{k-2}, q_k, q_{k-1}, q_{k-2}$ ; which is impossible since  $d_G(v, p_k), d_G(v, q_k) \geq 7$ . Thus  $j < k$ . From the maximality of  $j$ , it follows that  $d_G(r_j, u) = 2$ , and none of  $r_j, \dots, r_k$  equals or is adjacent to any vertex in  $P \cup Q \cup M$ . From the symmetry we may assume that  $u \in V(Q) \cup (V(M) \setminus V(P \cup Q))$ . Let  $w$  be a vertex adjacent to both  $u, r_j$ . If  $u \in L_k \cup L_{k-1}$  then  $k-j \leq 3$ , and so  $d_G(v, q_k) \leq 6$ , a contradiction; and if  $u \notin L_k \cup L_{k-1}$  and  $w \in L_k \cup L_{k-1}$  then  $u = q_{k-2}$  and  $k-j \leq 2$ , and again  $d_G(v, q_k) \leq 6$ , a contradiction. So  $u, w \notin L_k \cup L_{k-1}$ . Now there is a path of  $M \cup Q$  between  $u$  and  $p_{h-1}$ . If  $u \notin V(Q)$  then this path has length at most three, and its union with the path  $r_{k-1}-r_{k-2}-\dots-r_j-w-u$  is of length at most  $k-1-j+5 \leq k-h+5$ , since  $j \geq h-1$ , and so  $r_{k-1} \in X$  as required. If  $u \in V(Q)$ , then  $u$  is one of  $q_{j-2}, q_{j-1}, q_j, q_{j+1}, q_{j+2}$ , and so some path of  $M \cup Q$  between  $u$  and  $p_{h-1}$  has length at most  $(j+2) - (h+1) + 6$ , and its union with the path  $r_{k-1}-r_{k-2}-\dots-r_j-w-u$  has length at most

$$(j+2) - (h+1) + 6 + (k-1-j) + 2 = k - h + 8,$$

and again  $r_{k-1} \in X$ . This proves (1).

Now the set of vertices  $v \in L_k$  such that  $d_G(v, p_k) \leq 6$  or  $d_G(v, q_k) \leq 6$  has chromatic number at most  $2\kappa$ ; and since  $\chi(L_k) > 22\tau + 2\kappa$ , there exists a subset  $Z_0 \subseteq L_k$  with  $\chi(Z_0) > 22\tau$  such that

$d_G(v, p_k), d_G(v, q_k) \geq 7$  for each  $v \in Z_0$ . Every vertex in  $Z_0$  has a neighbour in  $X \cup Y$ , by (1); so we may assume that there exists  $Z_1 \subseteq Z_0$  with  $\chi(Z_1) > 11\tau$ , such that every vertex in  $Z_1$  is adjacent to a vertex in  $X$ . For each vertex  $x \in X$ , there is a path  $R$  as in the definition of  $X$ ; let  $R_x$  be a shortest such path. Then  $R_x$  has length at most  $k - h + 8$ , and at least  $(k - 1) - (h + 1)$ ; so there are eleven possibilities for its length, the numbers between  $k - h - 2$  and  $k - h + 8$ . For each  $c$  with  $k - h - 2 \leq c \leq k - h + 8$ , let  $X_c$  be the set of vertices  $x \in X$  such that  $R_x$  has length  $c$ . Then there exist  $c$  and  $Z_2 \subseteq Z_1$  with  $\chi(Z_2) \geq \chi(Z_1)/11 > \tau$ , such that every vertex in  $Z_2$  has a neighbour in  $X_c$ . Moreover we may choose  $Z_2$  such that  $G[Z_2]$  is connected. Let  $V$  be the union of the vertex sets of all the paths  $R_x$  ( $x \in X_c$ ). Note that  $V \subseteq L_0 \cup \dots \cup L_{k-1}$ . For  $0 \leq i \leq c$ , let  $V_i$  be the set of vertices  $u \in V$  such that the shortest path of  $G[V]$  between  $u, p_{h+1}$  has length  $i$ . Then  $(V_0, \dots, V_c)$  is a levelling. Moreover,  $V_c = X_c$ , and so no vertex in  $L_k$  has a neighbour in  $V_0, \dots, V_{c-1}$ . Define  $V_{c+1} = Z_2$ ; then also  $(V_0, \dots, V_{c+1})$  is a levelling.

Now no neighbour of  $p_{k-1}$  belongs to  $Z_0$ , and hence there are no edges between  $\{p_{h+2}, \dots, p_{k-1}\}$  and  $V_1 \cup \dots \cup V_{c+1}$ . Since  $G[L_k]$  is connected and  $p_{k-1}$  has a neighbour in  $L_k$ , there is a path  $G[L_k]$  between a vertex adjacent to  $p_{k-1}$  and a vertex with a neighbour in  $Z_2 = V_{c+1}$ . Choose a minimal such path,  $D$ , and let  $s$  be its end adjacent to  $p_{k-1}$ . Then  $(V_0, \dots, V_c, V_{c+1} \cup V(D), s)$  is a shower, since  $G[Z_2]$  is connected and hence so is  $G[V_{c+1} \cup V(D)]$ ; and its floor includes  $Z_2$  and hence has chromatic number more than  $\tau$ ; and  $p_{h+1}-p_{h+2}-\dots-p_{k-1}-s$  is a recirculator for it. This proves 5.3.  $\blacksquare$

Let  $\mathcal{S}$  be a shower with head  $z_0$  drain  $s$  and vertex set  $V$ . An induced path of  $G[V]$  between  $z_0, s$  is called a *jet* of  $\mathcal{S}$ . The set of all lengths of jets of  $\mathcal{S}$  is called the *jetset* of  $\mathcal{S}$ . If  $\mathcal{A}$  is a subset of the jetset of  $\mathcal{S}$ , then for each  $a \in \mathcal{A}$  there is a jet  $J_a$  with length  $a$ , and we say the set of jets  $\{J_a : a \in \mathcal{A}\}$  *realizes*  $\mathcal{A}$ . For  $\nu \geq 2$ , we say a shower  $\mathcal{S}$  is  $\nu$ -*complete* if there are  $\nu$  consecutive integers in its jetset, and  $\nu$ -*incomplete* otherwise. (Later we shall give a meaning to “1-complete”, but at this stage it is not needed.) We deduce:

**5.4** *Let  $\tau, \kappa \geq 0$  and  $\nu \geq 2$  be integers. Let  $G$  be a graph such that*

- $\chi^8(G) \leq \kappa$ ;
- $\chi(G) > 44\tau + 4\kappa$ ; and
- $G$  admits no hole  $\nu$ -interval.

*Then there is a  $\nu$ -incomplete shower in  $G$  with floor of chromatic number more than  $\tau$ .*

**Proof.** By 5.2 there is a levelling  $(L_0, \dots, L_k)$  with chromatic number more than  $22\tau + 2\kappa$ . By 5.3, there is a shower  $\mathcal{S}$ , with a recirculator, and with floor of chromatic number more than  $\tau$ . Since the union of the recirculator with any jet is a hole, and  $G$  admits no hole  $\nu$ -interval, it follows that  $\mathcal{S}$  is not  $\nu$ -complete. This proves 5.4.  $\blacksquare$

Thus, in order to prove 5.1, it suffices to show that if  $\nu, \kappa, G$  are as in the hypothesis of 5.1 then the floor of every  $\nu$ -incomplete shower in  $G$  has bounded chromatic number, and this is what we shall do.

## 6 Stabilizing a shower

A levelling  $(L_0, \dots, L_k)$  or shower  $(L_0, \dots, L_k, s)$  is *stable* if  $L_0, \dots, L_{k-1}$  are stable; and for  $\lambda \geq 0$  an integer, it is  $\lambda$ -*stable* if  $k \geq \lambda$  and  $L_i$  is stable for  $k - \lambda \leq i \leq k - 1$ . We would like to prove that there exists a stable shower (still with base of large  $\chi$ , but not as large as before), by converting the shower given by 5.4. This will take several steps. First we show how to convert a  $\nu$ -incomplete shower into a  $\nu$ -incomplete  $\lambda$ -stable shower (for any fixed  $\lambda$ ).

**6.1** *Let  $\tau, \lambda \geq 0$  and  $\nu \geq 2$  be integers, and let  $\mu = (\lambda + 1)(\nu - 1) + 1$ . Let  $G$  be a triangle-free graph, and let  $\mathcal{S}$  be a  $\nu$ -incomplete shower in  $G$ , with floor of chromatic number more than  $\nu\tau^{1+\mu}$ , and with levels  $L_0, \dots, L_k$ , where  $k \geq \mu$ . Then there is a  $\lambda$ -stable  $\nu$ -incomplete shower with floor of chromatic number more than  $\tau$ , and with levels  $L'_0, \dots, L'_h$ , such that  $0 \leq k - h \leq \mu - \lambda - 1$  and  $L'_i \subseteq L_i$  for  $0 \leq i \leq h$ .*

**Proof.** We may assume that for  $0 \leq i < k$ , every vertex in  $L_i$  has a neighbour in  $L_{i+1}$ ; for a vertex in  $L_i$  without this property could be deleted. Let  $z_0 \in L_0$ . For  $1 \leq j \leq \nu$ , let  $h_j = k - 1 - (\lambda + 1)(\nu - j)$ ; and for  $1 \leq j < \nu$ , let  $I_j = \{i : h_j < i < h_{j+1}\}$ . (Thus the sets  $I_j$  have cardinality  $\lambda$ , and there is an integer  $h_j$  between  $I_{j-1}$  and  $I_j$  that belongs to neither, that we use as insulation). For  $1 \leq j \leq \nu$ , let  $T_j$  be the set of vertices  $v \in L_{h_j}$  such that there are  $j$  induced paths between  $v$  and  $z_0$ , each with interior in  $L_1 \cup \dots \cup L_{h_{j-1}}$ , of lengths  $h_j, h_j + 1, \dots, h_j + j - 1$ .

(1)  $T_\nu = \emptyset$ .

Because suppose that  $v \in T_\nu$ . Then there are  $\nu$  induced paths between  $v$  and  $z_0$ , each with interior in  $L_1 \cup \dots \cup L_{k-2}$ , of lengths  $k - 1, k, \dots, k + \nu - 2$ , say  $R_1, \dots, R_\nu$ . Let  $s$  be the drain of  $\mathcal{S}$ ; and choose a minimal path  $Q$  between  $s, v$  with interior in  $L_k$ . Then for  $1 \leq i \leq \nu$ , the union of  $Q$  and  $R_i$  is a jet, contradicting that the shower is  $\nu$ -incomplete. This proves (1).

If  $X \subseteq L_0 \cup \dots \cup L_k$ , we denote by  $\theta(X)$  the chromatic number of the set of all descendants in  $L_k$  of members of  $X$ . Since  $T_1 = L_{h_1}$  it follows that

$$\theta(T_1) > \nu\tau^{1+\mu} \geq \tau^{k+1-h_2} + \tau^{k+1-h_3} + \dots + \tau^{k+1-h_\nu},$$

and so there exists  $j \in \{1, \dots, \nu\}$  maximum such that

$$\theta(T_j) > \tau^{k+1-h_{j+1}} + \tau^{k+1-h_{j+2}} + \dots + \tau^{k+1-h_\nu};$$

and  $j < \nu$  by (1). From the maximality of  $j$  it follows that  $\theta(T_j) - \theta(T_{j+1}) > \tau^{k+1-h_{j+1}}$ . Let  $S_{j+1}$  be the set of vertices in  $L_{h_{j+1}} \setminus T_{j+1}$  that have ancestors in  $T_j$ . For  $h_j < i < h_{j+1}$  let  $M_i$  be the set of vertices in  $L_i$  with an ancestor in  $T_j$  and a descendant in  $S_{j+1}$ .

(2)  $M_i$  is stable for  $h_j < i < h_{j+1}$ .

For suppose that  $x, y \in M_i$  are adjacent. Since  $G$  is triangle-free,  $x, y$  have no common parents and no common children. Let  $x', y' \in T_j$  be ancestors of  $x, y$  respectively (possibly equal). Let  $z \in S_{j+1}$  be a descendant of  $x$ . Now there are induced paths from  $y'$  to  $z_0$  with interior in  $L_1 \cup \dots \cup L_{h_{j-1}}$ ,

of lengths  $h_j, h_j + 1, \dots, h_j + j - 1$ . For each of these paths, its union with a path of length  $i - h_j$  between  $y$  and  $y'$ , a path of length  $h_{j+1} - i$  between  $z$  and  $x$ , and the edge  $xy$ , makes an induced path between  $z, z_0$ , of lengths  $h_{j+1} + 1, \dots, h_{j+1} + j$ . But also there is an induced path between  $z, z_0$  of length  $h_{j+1}$ , since  $z \in L_{h_{j+1}}$ ; and so  $z \in T_{j+1}$ , a contradiction. This proves (2).

Now every vertex in  $L_k$  with an ancestor in  $T_j$  has an ancestor in  $S_{j+1} \cup T_{j+1}$ . Since  $\theta(T_j) - \theta(T_{j+1}) > \tau^{k+1-h_{j+1}}$ , it follows that  $\theta(S_{j+1}) > \tau^{k+1-h_{j+1}}$ . By setting  $h = h_{j+1}$  and  $M_h = S_{j+1}$ , we have shown that:

(3) *There exist  $h$  with  $0 \leq k - h \leq \mu - \lambda - 1$ , and subsets  $M_i \subseteq L_i$  for  $h - \lambda \leq i \leq h$ , with the following properties:*

- $\theta(M_h) > \tau^{k+1-h}$ ;
- $M_i$  is stable for  $h - \lambda \leq i < h$ ; and
- every vertex in  $M_{i+1}$  has a neighbour in  $M_i$  for  $h - \lambda \leq i < h$ .

Choose such a value of  $h$ , maximal. Suppose first that  $\chi(M_h) \leq \tau$ . Since

$$\theta(M_h) > \tau^{k-h+1} \geq \tau \geq \chi(M_h)$$

it follows that  $h \neq k$ . Take a partition of  $M_h$  into  $\tau$  stable sets; then for one of these sets, say  $M'_h$ ,  $\theta(M'_h) \geq \theta(M_h)/\tau > \tau^{k-h}$ . Let  $M_{h+1}$  be the set of vertices in  $L_{h+1}$  with a neighbour in  $M_h$ ; then  $\theta(M_{h+1}) = \theta(M'_h) > \tau^{k-h}$ , contrary to the maximality of  $h$ . This proves that  $\chi(M_h) > \tau$ .

Let  $Z = L_h \cup \dots \cup L_k$ ; then  $G[Z]$  is connected since  $G[L_k]$  is connected and for  $0 \leq i < k$ , every vertex in  $L_i$  has a neighbour in  $L_{i+1}$ . Consequently

$$(L_0, \dots, L_{h-\lambda-1}, M_{h-\lambda}, \dots, M_{h-1}, Z, s)$$

is a shower  $\mathcal{S}'$  say. Its floor includes  $M_h$  and so has chromatic number more than  $\tau$ . Moreover, every jet for  $\mathcal{S}'$  is also a jet for  $\mathcal{S}$ ; and so  $\mathcal{S}'$  is  $\nu$ -incomplete. This proves 6.1. ■

## 7 U-bends

For  $\nu \geq 2$ , a shower  $(L_0, \dots, L_k, s)$  is a  $\nu$ -sprinkler if

- $G[L_k]$  is a path with one end  $s$  and with at least  $\nu$  vertices; let its vertices be  $v_1 - \dots - v_n$  in order, where  $v_1 = s$  and  $n \geq \nu$ ;
- for  $i = 1, \dots, n - \nu$ , no vertex in  $L_{k-1}$  is adjacent to  $v_i$ ; and
- for  $i = n - \nu + 1, \dots, n$ , some vertex in  $L_{k-1}$  is adjacent to  $v_i$  and to no other vertex in  $L_k$ .

Every  $\nu$ -sprinkler is therefore  $\nu$ -complete. We call  $\{v_i : n - \nu + 1 \leq i \leq n\}$  its *floor*.

We need another object, a “u-bend”, which is not exactly a shower; and also something which is partway to a u-bend, which we call a “w-bend”. We start with the latter. Let  $(L_0, \dots, L_k)$  be a levelling in  $G$  with vertex set  $V$ , and let  $U$  be an induced path of  $G$ . Suppose that

- $G[L_k]$  is an induced path;
- $V \cap V(U) = \emptyset$ ;
- $U$  has ends  $w, s$ , and there is at least one vertex in  $L_{k-1}$  adjacent to  $w$  and to a vertex in  $L_k$ ; and
- there are no edges between  $V(U)$  and  $L_k$ , and no vertex in  $L_{k-1}$  has a neighbour in  $L_k$  and a neighbour in  $V(U) \setminus \{w\}$ .

In this case, we call  $(L_0, \dots, L_k, U)$  a *w-bend*, and call  $s$  its *drain*; and any induced path of  $G[V \cup V(U)]$  between the vertex in  $L_0$  and the drain is called a *jet* of the w-bend. We call  $L_k$  its *floor*. (Since  $(L_0, \dots, L_k)$  is a levelling, every vertex in  $L_k$  has a neighbour in  $L_{k-1}$ .) Let  $G[L_k]$  have ends  $v_1, v_2$ ; then  $d_G(v_1, v_2)$  is called the *size* of the w-bend. If in addition:

- $w$  has a unique neighbour in  $L_{k-1}$ , say  $v$ ;
- $v$  has a unique neighbour in  $L_k$ , and this neighbour is an end of the path  $G[L_k]$ ; and
- every vertex in  $L_{k-1}$  has a neighbour in  $L_k$ ;

then we call  $(L_0, \dots, L_k, U)$  a *u-bend*. We need a containment relation for these objects:

- Let  $\mathcal{S} = (L_0, \dots, L_k, s)$  and  $\mathcal{S}' = (L'_0, \dots, L'_k, s')$  be showers. We say that  $\mathcal{S}'$  is *contained in*  $\mathcal{S}$  if they have the same drain, and  $L'_i \subseteq L_i$  for  $0 \leq i \leq k$ .
- Let  $\mathcal{S} = (L_0, \dots, L_k, s)$  be a shower, and let  $\mathcal{S}' = (L'_0, \dots, L'_k, U)$  be a w-bend. We say that  $\mathcal{S}'$  is *contained in*  $\mathcal{S}$  if they have the same drain, and  $L'_i \subseteq L_i$  for  $0 \leq i \leq k$ , and  $V(U) \subseteq L_k$ .
- Let  $\mathcal{S} = (L_0, \dots, L_k, W)$  be a w-bend, and let  $\mathcal{S}' = (L'_0, \dots, L'_k, U)$  be a u-bend. We say that  $\mathcal{S}'$  is *contained in*  $\mathcal{S}$  if they have the same drain, and  $L'_i \subseteq L_i$  for  $0 \leq i \leq k$ , and  $V(U) \subseteq L_k \cup V(W)$ .

In all three cases, every jet of  $\mathcal{S}'$  is a jet of  $\mathcal{S}$ .

We need to show that certain showers contain u-bends, and it is easier to show that they contain w-bends. Let us see first that that is enough, because a w-bend contains a u-bend (and containment is clearly transitive).

**7.1** *Let  $(L_0, \dots, L_k, W)$  be a w-bend in a triangle-free graph  $G$ , with size at least  $2p + 4$ . Then it contains a u-bend with size at least  $p$ .*

**Proof.** Let  $W$  have ends  $w, s$  where  $s$  is the drain. Let  $G[L_k]$  have vertices  $v_0, \dots, v_n$  say, in order. Since  $d_G(v_0, v_n) \geq 2p + 4$ , we may assume by exchanging  $v_0, v_n$  if necessary that  $d_G(w, v_0) \geq p + 2$ . Let  $Y$  be the set of vertices in  $L_{k-1}$  adjacent to  $w$  and to a vertex in  $L_k$ . By hypothesis,  $Y \neq \emptyset$ . Choose  $i \leq n$  minimum such that  $v_i$  has a neighbour in  $Y$ , say  $v$ . Since  $d_G(w, v_0) \geq p + 2$ , and  $d_G(w, v_i) = 2$ , it follows that  $d_G(v_i, v_0) \geq p$ . Let  $L'_{k-1}$  consist of all vertices in  $L_{k-1}$  with a neighbour in  $\{v_0, \dots, v_{i-1}\}$ , together with  $v$ . Then  $v$  is the unique neighbour of  $w$  in  $L'_{k-1}$ ; and so

$$(L_0, \dots, L_{k-2}, L'_{k-1}, \{v_0, \dots, v_i\}, W)$$

is a u-bend contained in  $(L_0, \dots, L_k, W)$ , and its size is at least  $p$ . This proves 7.1. ■

**7.2** Let  $\nu \geq 2$  be an integer, and let  $\mu \geq 1$ . Let  $\mathcal{S}$  be a shower in a triangle-free graph  $G$ . Let  $P$  be an induced path of  $G$  with  $V(P)$  a subset of the floor of  $\mathcal{S}$ , with ends  $w_1, w_2$  such that  $d_G(w_1, w_2) \geq 2(\mu + \nu)$ . Then  $\mathcal{S}$  contains either:

- a  $\nu$ -sprinkler with floor a subset of  $V(P)$ , or
- a  $u$ -bend with size at least  $\mu$  and with floor a subset of  $V(P)$ .

**Proof.** Let  $\mathcal{S} = (L_0, \dots, L_k, s)$ , and let  $L_{k-1}^1$  be the set of vertices in  $L_{k-1}$  with a neighbour in  $V(P)$ . If  $s \in V(P)$ , let  $u = s$  and let  $D$  be the one-vertex path with vertex  $s$ . If  $s \notin V(P)$ , then since  $G[L_k]$  is connected, there is an induced path  $D$  of  $G[L_k]$  between  $s$  and a vertex with a neighbour in  $V(P)$ ; choose a minimal such path  $D$ , with ends  $s, u$  say. From the minimality of  $D$ , no vertex in  $D \setminus \{u\}$  has a neighbour in  $V(P)$ .

Suppose that some vertex of  $D \setminus \{u\}$  has a neighbour in  $L_{k-1}^1$ ; and choose such a vertex,  $w$  say, such that the subpath  $D'$  of  $D$  between  $w, s$  is minimal. Then

$$(L_0, \dots, L_{k-2}, L_{k-1}^1, V(P), D')$$

is a  $w$ -bend contained in  $\mathcal{S}$ , of size at least  $2(\mu + 2)$  (since  $\nu \geq 2$ ), and the result follows from 7.1. We may therefore assume that there are no edges between  $D \setminus \{u\}$  and  $L_{k-1}^1$ .

Let  $Y$  be the set of vertices in  $L_{k-1}^1$  that are adjacent to  $u$ . Now no vertex of  $D$  except possibly  $u$  has a neighbour in  $L_{k-1}^1$ ; and  $u$  has at least one neighbour in  $V(P) \cup Y$ . Let  $G[L_k]$  have vertices  $v_0, \dots, v_n$  in order. By hypothesis,  $d_G(v_0, v_n) \geq 2(\mu + \nu)$ , so by exchanging  $v_0, v_n$  if necessary, we may assume that  $d_G(u, v_0) \geq \mu + \nu$ . Choose  $i$  minimum such that  $v_i$  has a neighbour in  $Y \cup \{u\}$ .

Suppose first that  $v_i$  has a neighbour in  $Y$ . Choose such a neighbour  $v$  say, and let  $L_{k-1}^2$  be the set of vertices in  $L_{k-1}$  with a neighbour in  $\{v_0, \dots, v_{i-1}\}$ , together with  $v$ . Now  $v_i$  is not adjacent to  $u$  (since  $G$  is triangle-free); and  $d_G(v_0, v_i) \geq d_G(v_0, u) - 2 \geq \mu$ ; so

$$(L_0, \dots, L_{k-2}, L_{k-1}^2, \{v_0, \dots, v_i\}, D)$$

is a  $u$ -bend contained in  $\mathcal{S}$  with size at least  $\mu$ , as required.

We may assume then that  $v_i$  has no neighbour in  $Y$ , and therefore  $v_i$  is adjacent to  $u$ . In summary, no vertex in  $L_{k-1}$  has a neighbour in  $V(D)$  and a neighbour in  $\{v_0, \dots, v_i\}$ ; and there are no edges between  $V(D)$  and  $\{v_0, \dots, v_i\}$  except the edge  $uv_i$ . Since  $d_G(v_0, u) \geq \mu + \nu$ , it follows that  $i \geq \mu + \nu - 1$ , and so  $i - \nu + 1 \geq \mu$ .

Suppose next that there exists a vertex in  $L_{k-1}$  adjacent to at least two of  $v_{i-\nu+1}, \dots, v_i$ . Choose  $j$  with  $i - \nu + 3 \leq j \leq i$  maximum such that some vertex in  $L_{k-1}$  is adjacent to  $v_j$  and to one of  $v_0, \dots, v_{j-2}$ ; choose  $h$  with  $0 \leq h \leq j - 2$  minimum such that some vertex in  $L_{k-1}$  is adjacent to  $v_h, v_j$ ; and choose  $v \in L_{k-1}$  adjacent to  $v_h, v_j$ . Let  $L_{k-1}^3$  be the set of vertices in  $L_{k-1}$  with a neighbour in  $\{v_0, \dots, v_{h-1}\}$ , together with  $v$ . Then since there is a path between  $u, v_h$  (via  $v$ ) of length  $j - i + 3 \leq \nu$ , it follows that  $d_G(u, v_h) \leq \nu$ , and so

$$d_G(v_h, v_0) \geq d_G(u, v_0) - \nu \geq \mu.$$

Let  $D_2$  be the path formed by the union of  $D$  and the path  $u-v_i-\dots-v_j$ . Then

$$(L_0, \dots, L_{k-2}, L_{k-1}^3, \{v_0, \dots, v_h\}, D_2)$$



is a u-bend contained in  $\mathcal{S}$ , of size at least  $\mu$ , as required.

We may therefore assume that no vertex in  $L_{k-1}$  is adjacent to more than one of  $v_{i-\nu+1}, \dots, v_i$ . Let  $L_{k-1}^4$  be the set of vertices in  $L_{k-1}$  with a neighbour in  $\{v_{i-\nu+1}, \dots, v_i\}$ . Every vertex in  $\{v_{i-\nu+1}, \dots, v_i\}$  has a neighbour in  $L_{k-1}^4$ , and  $u$  has no neighbour in  $L_{k-1}^4$ , so

$$(L_0, \dots, L_{k-2}, L_{k-1}^4, V(D) \cup \{v_{i-\nu+1}, \dots, v_i\}, s)$$

is a  $\nu$ -sprinkler contained in  $\mathcal{S}$ . This proves 7.2. ■

## 8 Jets of a shower

Let  $L_0, \dots, L_k$  be the levels of a shower or w-bend, and let  $J$  be a jet. Then at least one vertex of  $J$  belongs to  $L_{k-1}$ ; and we define the *tail* of  $J$  to be the minimal subpath of  $J$  between  $L_{k-1}$  and the drain. For  $\lambda \geq 0$ , we say that  $J$  is  $\lambda$ -*monotone* if  $\lambda \leq k$ , and  $J$  contains exactly one vertex of  $L_i$  for  $0 \leq i < k - \lambda$ . In every jet  $J$ , at least  $k - 1$  edges do not belong to its tail and have an end not in  $L_k$ . We say the *waste* of  $J$  is  $\mu$  if there are  $k - 1 + \mu$  edges of  $J$  that do not belong to its tail and have an end not in  $L_k$ ; and  $J$  is  $\mu$ -*wasteful* if its waste is at most  $\mu$ . Thus the waste is nonnegative.

A set of integers  $\mathcal{A}$  is *dense* if for all  $a_1, a_2 \in \mathcal{A}$  with  $a_1 < a_2$ , there does not exist  $b$  with  $a_1 < b < a_2$  such that  $b, b + 1 \notin \mathcal{A}$ ; that is, there are no two consecutive numbers both missing from  $\mathcal{A}$  between the first and last members of  $\mathcal{A}$ . If  $\mathcal{A}, \mathcal{B}$  are sets of integers, we define  $\mathcal{A} + \mathcal{B} = \{a + b : a \in \mathcal{A}, b \in \mathcal{B}\}$ . Thus if  $\mathcal{A}$  is dense, then for any integer  $t$ ,  $\mathcal{A} + \{t, t + 1\}$  is a set of consecutive integers of cardinality at least  $|\mathcal{A}| + 1$ .

Any subset of the floor of a shower is called a *mat*; and for a w-bend, we define its floor to be its only mat. The *size* of a mat  $M$  is the maximum of  $d_G(w_1, w_2)$  over all pairs  $w_1, w_2$  of vertices in the same component of  $G[M]$ . If  $M$  is a mat for a shower or w-bend  $\mathcal{S}$ , a jet  $J$  is an  $M$ -*jet* if there is no edge of  $J$  with an end in  $L_k \setminus M$  and an end in  $L_{k-1}$ . We define the  $M$ -*jetset* as the set of all lengths of  $M$ -jets. A w-bend  $(L_0, \dots, L_k, U)$  is  $\lambda$ -*stable* if  $k \geq \lambda$  and  $L_i$  is stable for  $k - \lambda \leq i \leq k - 1$ . In this section we prove the following.

**8.1** *Let  $\nu \geq 2$  be an integer, and let  $G$  be a triangle-free graph. If  $\mathcal{S}$  is a  $\nu$ -stable shower or w-bend in  $G$ , and  $M$  is a mat for  $\mathcal{S}$  of size at least  $3^{\nu+2}$ , then there is a set  $\mathcal{A}$  of integers, realized by a set of  $(\nu + 1)$ -monotone,  $3\nu^2$ -wasteful  $M$ -jets, such that  $|\mathcal{A}| \leq \nu + 1$ , and  $\mathcal{A}$  includes a dense subset of cardinality  $\nu$ , and there are two members of  $\mathcal{A}$  that differ by 1 or 3.*

**Proof.** We proceed by induction on  $\nu$ . Thus we assume that either  $\nu = 2$  or the result holds for  $\nu - 1$ . We claim we may assume:

(1) *There is a u-bend  $\mathcal{S}_1 = (L_0, \dots, L_k, U)$  contained in  $\mathcal{S}$ , and with  $L_k \subseteq M$ , of size at least  $3^{\nu+2}/2 - \nu$ .*

Assume first that  $\mathcal{S}$  is a  $\nu$ -stable shower in  $G$ , and  $M$  is a mat of size at least  $3^{\nu+2}$ . Let  $P$  be an induced path of  $G[M]$  with ends  $w_1, w_2$ , where  $d_G(w_1, w_2) \geq 3^{\nu+2}$ . If  $\mathcal{S}$  contains a  $\nu$ -sprinkler with floor a subset of  $V(P)$ , then the theorem holds, so we assume not. By 7.2 with  $\mu = 3^{\nu+2}/2 - \nu$ , it follows that  $\mathcal{S}$  contains a u-bend as in the claim. Next we assume that  $\mathcal{S}$  is a w-bend, of size at least  $3^{\nu+2}$ ; then the claim follows from 7.1. This proves (1).

Let  $\mathcal{S}_1 = (L_0, \dots, L_k, U)$  as in (1). Let  $U$  have ends  $u, s$  where  $s$  is the drain. Let  $q_0$  be the unique neighbour of  $u$  in  $L_{k-1}$ ; and let  $D$  be the path formed by adding the edge  $uq_0$  to  $U$ . There is an induced path  $q_0-q_1-\dots-q_n$  such that  $\{q_1, \dots, q_n\} = L_k$ ; and every vertex in  $L_{k-1}$  has a neighbour in  $L_k$ . Also,  $d_G(q_1, q_n) \geq 3^{\nu+2}/2 - \nu$ , and so  $d_G(q_0, q_n) \geq 3^{\nu+2}/2 - \nu - 1$ . We may assume that for  $0 \leq i \leq k-1$  every vertex in  $L_i$  has a neighbour in  $L_{i+1}$  (because any other vertex could be removed). Let  $V = L_0 \cup \dots \cup L_k$ .

We recall that for  $v \in V$ , its *height*  $h(v) = k - i$  where  $v \in L_i$ ; and we define the *reach* of  $v$  to be the maximum  $i \geq 1$  such that  $q_i$  is a descendant of  $v$ . (Since every vertex in  $V$  has a descendant in  $V(Q) \setminus \{q_0\}$ , this is well-defined.) Next we show that we may assume that:

(2) For  $1 \leq m \leq n$  there do not exist induced paths  $R_1, R_2$  of  $G[V]$  between  $q_0$  and  $q_m$  with the following properties:

- $|E(R_1)| + 1 = |E(R_2)| \leq 2\nu + 2$ ; and
- for all  $j$  with  $m < j \leq n$ ,  $q_j$  has no neighbour in  $V(R_1 \cup R_2) \setminus \{q_m\}$ .

For suppose that such  $m, R_1, R_2$  exist. Since  $R_1, R_2$  both have length at most  $2\nu + 2$  and have ends in  $L_k$  and  $L_{k-1}$ , it follows that every vertex of  $R_1 \cup R_2$  has height at most  $\nu + 1$ . Indeed, if  $y \in V(R_1 \cup R_2)$  then there is a subpath of one of  $R_1, R_2$  between  $y$  and  $q_m$ , which must have length at least  $h(y)$ , and since  $R_1, R_2$  both have length at most  $2\nu + 2$ , it follows that  $d_G(y, q_0) \leq 2\nu + 2 - h(y)$ . Consequently, if  $x \in V$  has a neighbour (say  $y$ ) in  $R_1 \cup R_2$  then

$$d_G(x, q_0) \leq d_G(y, q_0) + 1 \leq 2\nu - h(y) + 3 \leq 2\nu - h(x) + 4.$$

It follows that for every descendant in  $L_k$  of such a vertex  $x$ , its distance from  $q_0$  is at most  $d_G(x, q_0) + h(x) \leq 2\nu + 4$ . Since

$$d_G(q_0, q_n) \geq 3^{\nu+2}/2 - \nu - 1 > 2\nu + 4,$$

there exists  $m' < n$  such that  $d_G(q_0, q_{m'}) = 2\nu + 4$ , and  $d_G(q_0, q_j) > 2\nu + 4$  for all  $j$  with  $m' < j \leq n$ . Since  $q_{m+1}$  has a neighbour in  $R_1$ , it follows that  $d_G(q_{m+1}, q_0) \leq 2\nu + 4$ , and so  $m' \geq m + 1$ . For  $0 \leq i < k$  let  $L'_i$  be the set of all vertices in  $L_i$  with a descendant in  $\{q_j : m' < j \leq n\}$ . It follows that

$$(L'_0, \dots, L'_{k-1}, \{q_j : m \leq j \leq n\}, q_m)$$

is a shower  $\mathcal{S}'$  say. It is  $\nu$ -stable, since  $L'_i \subseteq L_i$  for  $0 \leq i < k$ . (It is not contained in  $\mathcal{S}$  since the drain is different.) Let its vertex set be  $V'$ . If  $v \in V' \setminus \{q_m\}$ , and  $v \in L_k$ , then  $v$  has no neighbour in  $V(R_1 \cup R_2) \setminus \{q_m\}$  from the properties of  $R_1, R_2$ ; and if  $v \notin L_k$ , then  $v$  has a descendant in  $\{q_j : m' < j \leq n\}$ , which therefore has distance in  $G$  more than  $2\nu + 4$  from  $q_0$ , and again  $v$  has no neighbour in  $R_1 \cup R_2$ . Thus there are no edges between  $V' \setminus \{q_m\}$  and  $V(R_1 \cup R_2)$  except the edge  $q_m q_{m+1}$ .

Now

$$d_G(q_n, q_{m'+1}) \geq d_G(q_n, q_0) - (2\nu + 5) \geq 3^{\nu+2}/2 - \nu - 1 - (2\nu + 5) \geq 3^{\nu+1}.$$

If  $\nu > 2$ , then from the inductive hypothesis on  $\nu$ , applied to  $\mathcal{S}'$  and the mat  $M' = \{q_{m'+1}, \dots, q_n\}$ , we deduce that there is a dense subset  $\mathcal{A}$  of the  $M'$ -jetset of  $\mathcal{S}'$  of cardinality  $\nu - 1$ , realized by a set of  $M'$ -jets of  $\mathcal{S}'$  that are  $\nu$ -monotone and  $3(\nu - 1)^2$ -wasteful. If  $\nu = 2$ , let  $\mathcal{A}$  be a singleton set containing the length of a 0-monotone, 0-wasteful  $M'$ -jet of  $\mathcal{S}'$ . In either case, let  $J$  be an  $M'$ -jet in

this set. Its tail has exactly one edge not in the path  $q_m - q_{m+1} - \dots - q_{m'}$ , and so at most  $3(\nu - 1)^2 + 1$  edges of  $J$  have an end not in  $L_k$ . Moreover, both  $J \cup R_1 \cup D$  and  $J \cup R_2 \cup D$  are jets of  $\mathcal{S}_1$ , and they are both  $(\nu + 1)$ -monotone (since every vertex of  $R_1 \cup R_2$  has height at most  $\nu + 1$ ). Since  $R_1, R_2$  have length at most  $2\nu + 2$ , it follows that these two jets both have waste at most at most  $3(\nu - 1)^2 + 1 + 2\nu + 2 \leq 3\nu^2$ . Let  $|E(R_1)| + |E(D)| = t$ ; then  $|E(R_2)| + |E(D)| = t + 1$ , so for each  $a \in \mathcal{A}$ , both  $a + t, a + t + 1$  belong to the jetset of  $\mathcal{S}_1$ , and so  $\mathcal{A} + \{t, t + 1\}$  is a subset of the jetset of  $\mathcal{S}_1$ , and hence of the  $M$ -jetset of  $\mathcal{S}$ , and this is a set of at least  $\nu$  consecutive integers. And this set is realized by  $M$ -jets of  $\mathcal{S}$  that are  $(\nu + 1)$ -monotone and have waste at most  $3\nu^2$ . Thus in this case the theorem holds. Consequently we may assume that no such  $m, R_1, R_2$  exist. This proves (2).

For each vertex  $v \in V$  with reach  $r < n$ , let  $f(v) \in V$  be defined as follows. There is a monotone path between  $v$  and  $q_r$ ; let  $X$  be the set of all vertices  $x$  such that  $x$  is adjacent to a vertex in a monotone path between  $v$  and  $q_r$ . Consequently  $q_{r+1} \in X$ , and so there exists  $x \in X$  with reach greater than  $r$ . Choose such a vertex  $x$  with maximum reach, and define  $f(v) = x$ . If  $v$  has reach  $n$  let  $f(v) = v$ .

Let  $v_1 = q_0$ , and for  $1 \leq i \leq \nu - 1$  let  $v_{i+1} = f(v_i)$ . We need to establish several properties of this sequence. Let  $t \leq \nu$  be maximum such that  $v_t \neq v_{t-1}$ . Thus either  $t = \nu$  or  $v_t$  has reach  $n$ . For  $1 \leq i \leq t$ ,  $r_i$  be the reach of  $v_i$ ; then  $r_1 = 1$ , and  $r_i < r_{i+1}$  for  $1 \leq i < t$ . For  $1 \leq i \leq t$  let  $P_i$  be a monotone path between  $v_i$  and  $q_{r_i}$  such that if  $i < t$  then  $v_{i+1}$  has a neighbour in  $P_i$ . The paths  $P_1, \dots, P_t$  are pairwise vertex-disjoint, because the reach of every vertex in  $P_i$  is precisely  $r_i$ , and  $r_1, \dots, r_t$  are all different. For  $1 \leq i < t$  let  $B_i$  be an induced path of  $G[V(P_i) \cup \{v_{i+1}\}]$  between  $v_i$  and  $v_{i+1}$ . Thus for  $1 \leq i \leq t$ ,  $B_1 \cup B_2 \cup \dots \cup B_{i-1} \cup P_i$  is a path, say  $C_i$ , between  $v_1$  and  $q_{r_i}$ . In particular,  $B_i$  has length at least one, so there is a unique vertex  $y_i$  of  $B_i$  adjacent to  $v_{i+1}$ . For  $1 \leq i \leq t$ , let  $\epsilon_i = 1$  if  $v_{i+1}, y_i \in L_k$ , and 2 otherwise.

(3)  $t = \nu$ ; for  $1 \leq i < \nu$ ,  $B_i$  has length  $h(v_i) - h(v_{i+1}) + \epsilon_i$ ; for  $1 \leq i \leq \nu$ ,  $C_i$  has length

$$1 + \sum_{1 \leq j < i} \epsilon_j;$$

and for  $1 \leq i \leq \nu$ ,  $C_i$  is an induced path.

Let  $1 \leq i < t$ . Since  $h(y_i) \leq h(v_i)$ , and  $h(v_{i+1}) \leq h(y_i) + 1$ , it follows that  $h(v_{i+1}) \leq h(v_i) + 1$ ; and since  $h(v_1) = 1$ , it follows inductively that  $h(v_i) \leq i$  for  $1 \leq i \leq t$ . Consequently for  $1 \leq i < t$ ,  $y_i$  has height at most  $\nu - 1$ ; and since the levelling is  $\nu$ -stable, it follows that  $y_i, v_{i+1}$  do not have the same height unless they both have height zero. Moreover,  $v_{i+1}$  is not a child of  $y_i$ , since the reach of  $v_{i+1}$  is greater than the reach of  $y_i$ ; so we have proved that either  $v_{i+1}$  is a parent of  $y_i$ , or  $v_{i+1}, y_i$  both have height zero. It follows that the length of  $B_i$  equals  $h(v_i) - h(v_{i+1}) + \epsilon_i$ , for all  $i < t$ .

For  $1 \leq i \leq t$ , the path  $B_1 \cup B_2 \cup \dots \cup B_{i-1}$  therefore has length

$$1 - h(v_i) + \sum_{1 \leq j < i} \epsilon_j,$$

and since  $P_i$  has length  $h(v_i)$ , it follows that  $C_i$  has length  $1 + \sum_{1 \leq j < i} \epsilon_j$ . Since this quantity is less than  $2\nu$ , and  $d_G(u, q_n) \geq 3^{\nu+2} > 2\nu$ , it follows that  $r_i < n$ . In particular,  $r_t < n$ , and so  $t = \nu$ .

We claim that for  $1 \leq i \leq \nu$ , the path  $C_i$  is induced; and prove this by induction on  $i$ . Certainly  $C_1$  is induced, so we may assume inductively that  $i < \nu$  and  $C_i$  is induced, and we prove that  $C_{i+1}$  is induced. Now  $C_{i+1}$  is obtained from a subpath of  $C_i$  by adding the edge  $y_i v_{i+1}$  and the path  $P_{i+1}$ ; so it suffices to check that there are no edges between  $B_1 \cup B_2 \cup \dots \cup B_i$  and  $P_{i+1}$  except the edge  $y_i v_{i+1}$ . Suppose then that  $y \in V(B_j)$  for some  $j \leq i$ , and  $x \in V(P_{i+1})$ , and  $xy$  is an edge. Since the reach of  $x$  equals  $r_{i+1}$ , it follows that  $x$  has no neighbour in any of  $P_1, \dots, P_{i-1}$ , and so  $y \in V(P_i)$ . Since also  $y \in V(B_j)$  for some  $j \leq i$ , it follows that  $y \in V(B_i \cap P_i)$ . Since  $B_i$  is induced and we may assume that  $(x, y) \neq (v_{i+1}, y_i)$ , it follows that  $x \neq v_{i+1}$ , and so  $h(v_{i+1}) > 0$  and  $h(x) < h(v_{i+1})$ . Since  $h(v_{i+1}) > 0$ , also  $v_{i+1}$  is a parent of  $y_i$ , and so  $h(x) \leq h(y_i)$ . But  $h(y) \geq h(y_i)$ , and since the levelling is  $\nu$ -stable and  $xy$  is an edge, it follows that  $y$  is a parent of  $x$ . But this is impossible since the reach of  $x$  is greater than the reach of  $y$ . This proves that each  $C_i$  is induced, and so completes the proof of (3).

For  $1 \leq j \leq n$ , let  $A_j$  be a monotone path between  $q_j$  and the shower head  $z_0$ . Thus  $A_j$  has length  $k$ . For  $1 \leq i \leq \nu$ , the reach of every vertex in  $A_{r_i+1}$  is at least  $r_i + 1$ , and so is greater than the reach of every vertex in  $C_i$ ; and so there is a path  $J_i$  formed by the union of  $D$ ,  $C_i$ , the edge  $q_{r_i} q_{r_i+1}$ , and  $A_{r_i+1}$ .

(4) For  $1 \leq i \leq \nu$  the path  $J_i$  is induced.

Suppose that some  $J_t$  is not induced, where  $1 \leq t \leq \nu$ . Consequently some vertex  $x$  of  $A_{r_t+1}$  is adjacent to some vertex  $y$  of  $C_t$ , and  $(x, y) \neq (q_{r_t}, q_{r_t+1})$ . Choose such a pair  $x, y$  with  $x$  of minimum height. Since  $y$  has height at most  $\nu$ , it follows that  $h(x) \neq h(y)$ ; and  $x$  is not a child of  $y$  since the reach of  $x$  is greater than the reach of  $y$ . Thus  $x$  is a parent of  $y$ . Let  $y \in V(P_j)$  where  $j \leq t$ . Since  $x$  has a neighbour in  $P_j$ , it follows that the reach of  $x$  is at most  $r_{j+1}$ ; and so  $r_t < r_{j+1}$ . Consequently  $t < j + 1$ , and since  $j \leq t$  it follows that it follows that  $j = t$ , and so  $y \in V(P_t)$ . Let  $a$  be the vertex of  $A_{r_t+1}$  of height 1. Now there are two cases. First suppose that  $a$  is nonadjacent to  $q_j$  for  $r_t + 2 \leq j \leq n$ . Let  $i_1 = r_t + 1$ , and let  $R_1$  be the path formed by the union of  $C_t$  and the edge  $q_{r_t} q_{r_t+1}$ , and let  $R_2$  be the path formed by the union of the subpath of  $C_t$  between  $u, y$ , the edge  $xy$ , and the subpath of  $A_{r_t+1}$  between  $x, q_{r_t+1}$ . Note that  $R_1$  is induced by (2), and  $R_2$  is induced since we chose  $xy$  with  $x$  of minimum height. Also  $R_1$  has length at most  $2\nu$ , and  $R_2$  has length one more. This is therefore impossible by (2). Consequently there exists  $j > r_t + 1$  adjacent to  $a$ ; choose such a value of  $j$ , maximum, and let  $i_1 = j$ . Let  $R_2$  be the path formed by the union of  $C_t$  and the path  $q_{r_t} - q_{r_t+1} - a - q_j$ , and let  $R_1$  be the path formed by the union of the subpath of  $C_t$  between  $u, y$ , the edge  $xy$ , the subpath of  $A_{r_t+1}$  between  $x, a$ , and the edge  $aq_j$ . In this case  $R_2$  has length at most  $2\nu + 2$ , and  $R_1$  has length one less. Since  $j \geq r_t + 3$  (because  $G$  is triangle-free) it follows that both paths are induced, and again this contradicts (2). Thus there is no such  $t$ . This proves (4).

Since each  $J_i$  is induced, it is therefore a jet for the u-bend  $\mathcal{S}_1$  (and hence an  $M$ -jet for  $\mathcal{S}$ ), of length  $k + 2 + \sum_{1 \leq j < i} \epsilon_j + |V(D)|$ , and with tail the path  $D$ ; and since  $J_i \setminus V(D)$  has length at most  $k + 2\nu$ , and all vertices of  $B_i$  have height at most  $\nu$ , it follows that  $J_i$  is  $\nu$ -monotone and  $2\nu$ -wasteful (and hence  $3\nu^2$ -wasteful). The shortest of these jets is  $J_1$ , and it has length  $k + 1 + |V(D)|$ . Let  $A_0$  be a monotone path between  $v_1$  and  $z_0$ ; then also there is an  $M$ -jet formed by the union of  $D$ , the edge  $uv_1$ , and  $A_0$ , of length  $k - 2 + |V(D)|$  (so, three less than the length of  $J_1$ ). Consequently these  $M$ -jets realize a subset of the  $M$ -jetset satisfying the theorem. This proves 8.1.  $\blacksquare$

The previous result will have several applications later in the paper. First, let us use it to convert a  $\lambda$ -stable shower into a stable shower.

**8.2** *Let  $\kappa, \tau \geq 0$  and  $\nu \geq 2$  be integers, and let  $\rho = 3^{\nu+2}$ . Let  $G$  be a triangle-free graph such that  $G$  has no hole  $\nu$ -interval, and  $\chi^\rho(G) \leq \kappa$ . If  $G$  admits a  $\nu$ -incomplete  $(\nu + 2)$ -stable shower with floor of chromatic number more than  $\kappa + \tau$ , then  $G$  admits a  $\nu$ -incomplete stable shower with floor of chromatic number more than  $\tau$ .*

**Proof.** Let  $\mathcal{S}$  be a  $\nu$ -incomplete  $(\nu + 2)$ -stable shower  $(L_0, \dots, L_k, s)$  in  $G$ . Thus  $k \geq \nu + 2$ . Let  $j = k - \nu - 2$ ; then  $L_i$  is stable for  $j \leq i < k$ . Let  $L_0 = \{z_0\}$ . Let  $X$  be the set of all vertices  $v \in L_j$  such that there is an induced path  $P_v$  of  $G$  between  $v, z_0$  with length  $j + 1$ , such that every vertex in  $P_v$  different from  $v$  belongs to one of  $L_0, \dots, L_{j-1}$ . Let  $Y = L_j \setminus X$ . Let  $X'$  be the set of vertices in  $L_k$  with an ancestor in  $X$ , and  $Y'$  the set of vertices in  $L_k$  with an ancestor in  $Y$ . Thus  $X' \cup Y' = L_k$ .

Suppose that  $\chi(G[X']) > \kappa$ . For  $j \leq i \leq k - 1$ , let  $L'_i$  be the set of vertices in  $L_i$  with an ancestor in  $X$ . Then

$$(L_0, \dots, L_{j-1}, L'_j, \dots, L'_{k-1}, L_k, s)$$

is a  $\nu$ -stable shower  $\mathcal{S}_1$  say, and its floor includes  $X'$ . This is contained in  $\mathcal{S}$ , so  $\mathcal{S}_1$  is  $\nu$ -incomplete. Since  $\chi(C) > \kappa$ , there exist  $w_1, w_2 \in X'$ , in the same component of  $G[X']$ , with  $d_G(w_1, w_2) > \rho \geq 3^{\nu+2}$ . By 8.1 there is a dense subset  $\mathcal{A}$  of the jetset of  $\mathcal{S}_1$  of cardinality  $\nu$ , and a set  $\{J_a : a \in \mathcal{A}\}$  of  $(\nu + 1)$ -monotone jets for  $\mathcal{S}_1$  realizing  $\mathcal{A}$ . Thus for each  $a \in \mathcal{A}$ ,  $J_a$  contains exactly one vertex of  $L'_i$  for  $0 \leq i \leq j$ . In particular,  $J_a$  contains exactly one vertex in  $L'_j = X$ , say  $x$ . The subpath of  $J_a$  between  $x, z_0$  has length  $j$ , and so the subpath  $R_a$  say of  $J_a$  between  $x, s$  has length  $|E(J_a)| - j$ . By definition of  $X$ , the path  $P_x$  exists and has length  $j + 1$ ; and since both  $R_a, P_x$  have exactly one vertex in  $L_j$ , their union  $R_a \cup P_x$  is an induced path between  $s, z_0$  of length exactly one more than the length of  $J_a$ . Now both  $J_a$  and  $R_a \cup P_x$  are jets of  $\mathcal{S}_1$  and hence of  $\mathcal{S}$ . Thus  $\mathcal{A} + \{0, 1\}$  is a subset of the jetset of  $\mathcal{S}$ . But this set consist of at least  $\nu + 1$  consecutive integers, since  $\mathcal{A}$  is dense of cardinality  $\nu$ ; and this is impossible since  $\mathcal{S}$  is not  $\nu$ -complete. This proves that  $\chi(G[X']) \leq \kappa$ .

Consequently  $\chi(G[Y']) > \tau$ . For  $0 \leq i \leq j$ , let  $L'_i$  be the set of vertices in  $L_i$  with a descendant in  $Y$ , and for  $j + 1 \leq i \leq k$ , let  $L'_i$  be the set of vertices in  $L_i$  with an ancestor in  $Y$ . Then  $(L'_0, \dots, L'_{k-1}, L_k, s)$  is a shower  $\mathcal{S}'$  say, with floor of chromatic number more than  $\tau$  since its floor includes  $Y'$ . This is contained in  $\mathcal{S}$ , so  $\mathcal{S}'$  is  $\nu$ -incomplete. We claim that  $\mathcal{S}'$  is stable. For certainly  $L_j, \dots, L_{k-1}$  are stable, since  $\mathcal{S}$  is  $(\nu + 2)$ -stable. Suppose that  $0 \leq i \leq j - 1$  and  $y, y' \in L'_i$  are adjacent. Since  $y$  has a descendant in  $Y$ , there is a path between  $y$  and  $Y$  of length  $j - i$ ; and since  $y' \in L_i$ , there is a path between  $y', z_0$  of length  $i$ . Since  $G$  is triangle-free,  $yy'$  is the only edge between these two paths; and so their union, together with this edge, is an induced path between  $y, z_0$  of length  $j + 1$ , contradicting that  $y \notin X$ . This proves that  $\mathcal{S}'$  is stable; and so the theorem holds. This proves 8.2. ■

We deduce:

**8.3** *Let  $\tau \geq 0$  and  $\nu \geq 2$  be integers, and let  $\rho = 3^{\nu+2}$ . Let  $G$  be a triangle-free graph, such that  $G$  has no hole  $\nu$ -interval, and  $\chi^\rho(G) \leq \kappa$ . If  $\chi(G) > 44\nu(\kappa + \tau)^{(\nu+1)^2} + 4\kappa$ , then  $G$  admits a  $\nu$ -incomplete stable shower with floor of chromatic number more than  $\tau$ .*

**Proof.** By 5.4, there is a  $\nu$ -incomplete shower  $(L_0, \dots, L_k, s)$  in  $G$ , with floor of chromatic number more than  $\nu(\kappa + \tau)^{(\nu+1)^2}$ . Then  $k > \rho$ , since  $\nu(\kappa + \tau)^{(\nu+1)^2} \geq \kappa$ . Since  $\rho \geq (\nu + 3)(\nu - 1) + 1$ , 6.1 (with  $\lambda = \nu + 2$ ) implies that there is a  $(\nu + 2)$ -stable  $\nu$ -incomplete shower in  $G$  with floor of chromatic number more than  $\kappa + \tau$ , so the result follows from 8.2. This proves 8.3.  $\blacksquare$

The reason for controlling the waste of the jets that are output by 8.1 is that a jet with bounded waste can be covered by a bounded number of monotone paths. More precisely:

**8.4** *Let  $\mathcal{S} = (L_0, \dots, L_k, s)$  be a shower in a graph  $G$ , and let  $J$  be a  $\mu$ -wasteful jet of  $\mathcal{S}$ . Then there is a set of at most  $\mu + 1$  monotone paths of  $\mathcal{S}$  such that every vertex of  $J$  in  $L_0 \cup \dots \cup L_{k-1}$  belongs to one of these paths.*

**Proof.** Choose  $d \in V(J)$  such the tail  $T$  of  $J$  has ends  $d, s$ . Then no vertex of  $T$  belongs to  $L_0 \cup \dots \cup L_{k-1}$  except  $d$ . Let  $P$  be the subpath of  $J$  between  $z_0, d$ , where  $z_0 \in L_0$ . At most  $k - 1 + \mu$  edges of  $P$  have an end not in  $L_k$ , since the waste of  $J$  is at most  $\mu$ . Let us say the *height* of an edge  $uv$  of  $P$  is the maximum of the heights of  $u, v$ . Thus at most  $k - 1 + \mu$  edges of  $P$  have nonzero height. As  $P$  is traversed starting from  $d$ , the number of edges in it that have height at least 2 and different from the heights of all previous edges is at least  $k - 1$ , since the difference of the heights of  $z_0, d$  is  $k - 1$ ; and so there are at most  $\mu$  edges of  $P$  that have height 1 or the same nonzero height as some earlier edge. By removing all such edges, we decompose  $P$  into at most  $\mu + 1$  paths each of which is either monotone or a path of  $G[L_k]$ ; and every vertex of  $P$  in  $L_0 \cup \dots \cup L_{k-1}$  belongs to one of these monotone paths. This proves 8.4.  $\blacksquare$

## 9 Stable showers

From now on, there is no need to consider general showers; we might as well just concern ourselves with stable showers, in view of 8.3. To complete the proof of 5.1, we only need to show that if  $\nu, \kappa, G$  satisfy the hypotheses of 5.1 then every  $\nu$ -incomplete stable shower in  $G$  has floor with bounded  $\chi$ , and that is the goal of the remainder of the paper.

If  $\mathcal{S} = (L_0, \dots, L_k)$  is a levelling, and  $X \subseteq L_0 \cup \dots \cup L_k$ , we denote by  $\Theta(X)$  or  $\Theta_{\mathcal{S}}(X)$  the set of vertices in  $L_k$  that have an ancestor in  $X$ , and by  $\theta(X)$  or  $\theta_{\mathcal{S}}(X)$  the chromatic number of  $\Theta(X)$ . (We use the same notation if  $L_0, \dots, L_k$  are the levels of a shower.)

We are concerned with a triangle-free graph which admits no hole  $\nu$ -interval; and we will not need to use induction on  $\nu$  any more; so from now on we shall fix  $\nu \geq 2$ , to avoid having to carry it along. We might as well also set  $\rho = 3^{\nu+2} + 4$ , for the remainder of the paper, and fix  $\kappa \geq 0$ . Let us say a graph  $G$  is a *candidate* if  $G$  is triangle-free, and admits no hole  $\nu$ -interval, and  $\chi^{\rho}(G) \leq \kappa$ . Our eventual goal is to prove that every stable shower in every candidate has floor of bounded  $\chi$ .

Let  $\mathcal{S}$  be a stable shower, with vertex set  $V$ , and let  $M$  be a mat. For  $X \subseteq V$ , we denote the set of vertices in  $M$  with an ancestor in  $X$  by  $M(X)$ , or  $M_{\mathcal{S}}(X)$  if there is danger of ambiguity (and we write  $M(v)$  for  $M(\{v\})$ .)

We already defined “containment” for showers, but now we need a slightly different inclusion relation. Let  $\mathcal{S} = (L_0, \dots, L_k, s)$  be a stable shower, and let  $\mathcal{S}' = (L'_0, \dots, L'_{k'}, s')$  be a shower, both in a graph  $G$ . We say that  $\mathcal{S}'$  is a *subshower* of  $\mathcal{S}$  if

- $s = s'$
- $k' \leq k$ ; let  $h = k - k'$ ; and
- $L'_i \subseteq L_{i+h}$  for  $0 \leq i \leq k'$ .

In particular, let  $M$  be a mat for  $\mathcal{S}$ , and let  $z_1 \in L_h$ , where  $0 \leq h < k$ ; then we define the *subshower of  $\mathcal{S}$  under  $z_1$  and over  $M$*  to be  $(L'_h, \dots, L'_{k-1}, L_k, s)$ , where  $L'_i$  is the set of all descendants of  $z_1$  in  $L_i$  that have descendants in  $M$ .

If  $\mathcal{S} = (L_0, \dots, L_k, s)$  is a shower, we define its *union*  $U(\mathcal{S})$  to be  $L_0 \cup \dots \cup L_{k-1}$ . (Note that this is different from the vertex set, as we do not include  $L_k$ .)

**9.1** *Let  $G$  be a candidate. Let  $\mathcal{S} = (L_0, \dots, L_k, s)$  be a stable shower in  $G$ , and let  $z_1, z_2 \in U(\mathcal{S})$ , either equal or nonadjacent. For  $i = 1, 2$ , let  $\mathcal{S}_i$  be a subshower with union  $V_i$  and head  $z_i$  respectively, and let  $M_i$  be a mat for  $\mathcal{S}_i$ . Suppose that  $V_1 \cap V_2 = \{z_1\} \cap \{z_2\}$ , and  $\chi(M_1) > \kappa$ . Let  $X$  be the set of vertices in  $V_2 \setminus \{z_1\}$  with a neighbour in  $V_1 \setminus \{z_2\}$ ; and for every monotone path  $R$  in  $G[V_1]$  between  $z_1$  and  $M_1$ , let  $X(R)$  denote the set of vertices in  $V_2 \setminus \{z_1\}$  with a neighbour in  $V(R) \setminus \{z_2\}$ . Then either*

- $z_1 \neq z_2$ , and there are  $\nu$  induced paths  $Q_0, \dots, Q_{\nu-1}$  of  $G[V_1 \cup V_2 \cup L_k]$  between  $z_1, z_2$ , such that  $|E(Q_i)| = |E(Q_0)| + i$  for  $0 \leq i < \nu$ ; or
- $\chi(M_2 \setminus \Theta(X)) \leq 2\kappa$ , and for all  $\tau \geq 0$ , if  $\chi(M_2) > 2\kappa + (\nu + 1)(3\nu^2 + 1)\tau$ , then there is a monotone path  $R$  of  $G[V_1]$  between  $z_1$  and  $M_1$  such that  $\chi(M_2(X(R))) > \tau$ .

**Proof.** Choose a component of  $G[M_1]$  with maximum chromatic number; and since this chromatic number is larger than  $\kappa$ , it follows that there are two vertices of this component with distance more than  $\rho$ . Consequently there is a path  $P_1$  with  $V(P_1) \subseteq M_1$  joining two vertices with distance at least  $3^{\nu+2}$  (in  $G$ ). Choose a minimal such path  $P_1$ , and let  $w_1$  be one of its ends. From the minimality of  $P_1$  it follows that  $d_G(w_1, v) \leq 3^{\nu+2}$  for every vertex  $v$  of  $P_1$ .

Let  $C_2$  be a connected induced subgraph of  $G[M_2 \setminus N^\rho[w_1]]$  with  $\chi(C_2) > \kappa$  (if there is no such subgraph, then  $\chi(M_2) \leq 2\kappa$  and the theorem holds). In addition, choose  $C_2$  with  $V(C_2) \cap \Theta_{\mathcal{S}_2}(X) = \emptyset$  if possible. Every path of  $G$  between  $C_2$  and  $P_1$  has length at least 3, since  $\rho \geq 3^{\nu+2} + 3$ .

Let  $(L_{h_1}^1, \dots, L_{k-1}^1, L_k, s)$  be the subshower  $\mathcal{S}'_1$  of  $\mathcal{S}_1$  above  $P_1$ , and let its union be  $V'_1$ . Since  $G[L_k]$  is connected, there is a path of  $G[L_k]$  between  $V(P_1)$  and  $C_2$ ; let  $D$  be a minimal path of  $G[L_k]$  such that one end (say  $d_1$ ) has a neighbour in  $V(P_1) \cup L_{k-1}^1$  and the other (say  $d_2$ ) has a neighbour in  $C_2$ .

(1) *There is a set  $\mathcal{A}_1$  of integers, of cardinality at most  $\nu + 1$ , including a dense subset of cardinality  $\nu$ , and containing two integers  $x, y$  with  $y - x \in \{1, 3\}$ , such that the following holds. For each  $a \in \mathcal{A}_1$  there is an induced path  $J_a$  of  $G$  between  $d_1, z_1$  of length  $a$ , such that*

- $V(J_a) \subseteq V'_1 \cup \{d_1\}$ ; and
- *there is a set of  $3\nu^2 + 1$  monotone paths of  $G[V'_1]$  between  $V(P_1)$  and  $z_1$ , such that every vertex of  $V(J_a) \setminus (V(P_1) \cup \{d_1\})$  belongs to one of these paths.*

Let  $D_1$  be the one-vertex path with vertex  $d_1$ . If  $d_1$  has no neighbour in  $V(P_1)$ , then

$$(L_{h_1}^1, \dots, L_{k-1}^1, V(P_1), D_1)$$

is a w-bend  $\mathcal{S}'_1$  of size at least  $3^{\nu+2}$ ; and otherwise  $(L_{h_1}^1, \dots, L_{k-1}^1, V(P_1) \cup \{d_1\}, d_1)$  is a shower  $\mathcal{S}'_1$ . In either case we can apply 8.1 to  $\mathcal{S}'_1$ , and deduce that there is a subset  $\mathcal{A}_1$  of the jetset of  $\mathcal{S}'_1$ , of size at most  $\nu + 1$ , including a dense subset of cardinality  $\nu$ , and containing two integers  $x, y$  with  $y - x \in \{1, 3\}$ ; and realized by a set of jets of  $\mathcal{S}'_1$  that are  $3\nu^2$ -wasteful. By 8.4, this proves (1).

Since  $|\mathcal{A}_1| \leq \nu + 1$ , there is a set of at most  $(\nu + 1)(3\nu^2 + 1)$  monotone paths of  $G[V'_1]$  between  $V(P_1)$  and  $z_1$  such that, if  $Y$  denotes the set of vertices in these paths, then  $V(J_a) \subseteq Y \cup V(P_1) \cup \{d_1\}$  for each  $a \in \mathcal{A}_1$ . Let  $X'$  denote the set of vertices in  $V_2 \setminus \{z_1\}$  with a neighbour in  $Y \setminus \{z_2\}$ . Let  $h_2$  be such that  $z_2 \in L_{h_2}$ , and for  $h_2 \leq i \leq k$  let  $L_i^2$  be the set of vertices in  $L_i$  such that there is a monotone path of  $G[V_2 \setminus X']$  between  $v, z_2$ . It follows that no vertex in  $Y \setminus \{z_2\}$  has a neighbour in  $(L_{h_2}^2 \cup \dots \cup L_k^2) \setminus \{z_1\}$ .

(2) If  $\chi(L_k^2 \cap V(C_2)) > \kappa$  then the theorem holds.

For then there exists an induced path  $P_2$  of  $G[L_k^2 \cap V(C_2)]$  with ends at distance at least  $\rho$ . Since  $d_2$  has a neighbour in  $C_2$ , it follows that  $G[V(C_2) \cup V(D)]$  is connected. Thus

$$(\{z_1\}, L_{h+1}^2, \dots, L_{k-1}^2, V(C_2) \cup V(D), d_1)$$

is a shower  $\mathcal{S}'_2$ , and  $L_k^2 \cap V(C_2)$  is a mat  $M$  say; and by 8.1, there is a dense subset  $\mathcal{A}_2$  of the  $M$ -jetset of  $\mathcal{S}'_2$  of cardinality  $\nu$ . We claim that  $\mathcal{A}_1 + \mathcal{A}_2$  contains a set  $\mathcal{B}$  of  $\nu$  consecutive integers. To see this, suppose first that there are two consecutive integers  $a, a + 1 \in \mathcal{A}_2$ . Let  $\mathcal{A}'$  be a dense subset of  $\mathcal{A}_1$  of cardinality  $\nu$ ; then  $\mathcal{A}' + \{a, a + 1\}$  consists of at least  $\nu + 1$  consecutive integers, all contained in  $\mathcal{A}_1 + \mathcal{A}_2$  as required. We may assume that no two members of  $\mathcal{A}_2$  are consecutive. Since  $\mathcal{A}_2$  is dense of cardinality  $\nu$ , there exists  $s$  such that  $s, s + 2, s + 4, \dots, s + 2(\nu - 1) \in \mathcal{A}_2$ . But there exist  $x, y \in \mathcal{A}_1$  with  $y - x \in \{1, 3\}$ ; and then

$$\{s, s + 2, s + 4, \dots, s + 2(\nu - 1)\} + \{x, y\}$$

contains  $\nu$  consecutive integers (indeed, almost  $2\nu$ ). This proves that  $\mathcal{B}$  exists.

If  $z_1 = z_2$  then for every  $J_a$  ( $a \in \mathcal{A}_1$ ) and every  $M$ -jet of  $\mathcal{S}'_2$ , their union is a hole; and so  $G$  has holes of every length in  $\mathcal{B}$ , and so has a hole  $\nu$ -interval, which is impossible since  $G$  is a candidate. Thus  $z_1 \neq z_2$ , and so they are nonadjacent; but then for every  $J_a$  ( $a \in \mathcal{A}_1$ ) and every  $M$ -jet of  $\mathcal{S}'_2$ , their union is an induced path between  $z_1, z_2$  and the theorem holds. This proves (2).

We may therefore assume that  $\chi(L_k^2 \cap V(C_2)) \leq \kappa$ . Consequently,  $V(C_2) \not\subseteq L_k^2$ , and therefore  $V(C_2) \cap \Theta_{\mathcal{S}_2}(X') \neq \emptyset$ . From the choice of  $C_2$ , it follows that  $\chi(M_2 \setminus \Theta(X)) \leq 2\kappa$  (for otherwise we would have chosen  $C_2$  with  $V(C_2) \subseteq M_2 \setminus (\Theta(X) \cup N^\rho[w_1])$ ). This proves the first statement of the theorem.

Now let  $\tau \geq 0$ , with  $\chi(M_2) > 2\kappa + (\nu + 1)(3\nu^2 + 2)\tau$ ; then we may choose  $C_2$  with  $\chi(C_2) > \kappa + (\nu + 1)(3\nu^2 + 2)\tau$ . Since  $\chi(C_2 \setminus \Theta_{\mathcal{S}_2}(X)) \leq \kappa$ , it follows that  $\chi(M_2(X)) > (\nu + 1)(3\nu^2 + 1)\tau$ . Thus, one of the  $(\nu + 1)(3\nu^2 + 1)$  monotone paths satisfies the theorem. This proves 9.1.  $\blacksquare$



There is a special case of 9.1 that we often need, and we extract it to make application easier.

**9.2** *Let  $G$  be a candidate. Let  $(L_0, \dots, L_k, s)$  be a stable shower in  $G$ , and let  $z_1 \in U(\mathcal{S})$ . Let  $A, B$  be disjoint sets of children of  $z_1$ . Let  $M$  be a mat for  $\mathcal{S}$ , and suppose that  $\chi(M(B) \setminus M(A)) > \kappa$ . Then  $\chi(M(A) \setminus M(B)) \leq 2\kappa$ , and for all  $\tau \geq 0$ , if  $\chi(M(A)) > 2\kappa + (\nu + 1)(3\nu^2 + 1)\tau$ , then there is a monotone path  $R$  between  $B, M(B) \setminus M(A)$  such that  $\chi(M(X)) > \tau$ , where  $X$  denotes the set of vertices with a parent in  $V(R)$  and an ancestor in  $A$  and a descendant in  $M(A)$ .*

**Proof.** Let  $\mathcal{S}_1$  be the maximal subshower with head  $z_1$  such that no vertex of its union has an ancestor in  $A$ , and every vertex of its union except  $z_1$  is a descendant of a member of  $B$ ; and let  $\mathcal{S}_2$  be the maximal subshower with head  $z_1$  such that every vertex of its union except  $z_1$  has an ancestor in  $A$ . Let their unions be  $V_1, V_2$  respectively. Then  $V_1 \cap V_2 = \{z_1\}$ , and no vertex in  $V_1$  has a parent in  $V_2$ . Also  $M(B) \setminus M(A)$  is a mat for  $\mathcal{S}_1$ , and  $M(A)$  for  $\mathcal{S}_2$ . By 9.1, the result follows. This proves 9.2.  $\blacksquare$

**9.3** *Let  $G$  be a candidate. Let  $(L_0, \dots, L_k, s)$  be a stable shower in  $G$ , with union  $V$ , let  $z_1 \in V$ , let  $Y$  be a subset of the set of children of  $z_1$ , and let  $M \subseteq \Theta(Y)$ . For  $\tau \geq 0$ , if*

$$\chi(M) > \tau + ((\nu + 1)(3\nu^2 + 1) + 7)\kappa,$$

*then there exists  $z_2 \in Y$  such that  $\chi(M(z_2)) > \tau$ .*

**Proof.** Choose  $A \subseteq Y$  minimal such that  $\chi(M(A)) > 2\kappa + \tau$ . Suppose that  $\chi(M(A)) > 3\kappa + \tau$ , and choose  $z_2 \in A$ ; then from the minimality of  $A$ ,  $\chi(M(A \setminus \{z_2\})) \leq 2\kappa + \tau$ , and so

$$\chi(M(z_2) \setminus M(A \setminus \{z_2\})) > \kappa.$$

By 9.2 applied to  $A \setminus \{z_2\}$  and  $\{z_2\}$ , it follows that  $\chi(M(A) \setminus M(z_2)) \leq 2\kappa$ ; and since  $\chi(M(A)) > 2\kappa + \tau$ , it follows that  $\chi(M(z_2)) > \tau$ , as required.

We may assume therefore that  $\chi(M(A)) \leq 3\kappa + \tau$ . Let  $n = (\nu + 1)(3\nu^2 + 1)\kappa$ ; then

$$\chi(M) > \tau + n + 7\kappa \geq \chi(M(A)) + n + 4\kappa,$$

so we may choose  $B \subseteq Y$  with  $A \subseteq B$ , minimal such that  $\chi(M(B)) > \chi(M(A)) + n + 2\kappa$ . Again, by the same argument, we may assume that

$$\chi(M(B)) \leq \chi(M(A)) + n + 3\kappa \leq \tau + n + 6\kappa;$$

and since  $\chi(M) > \tau + n + 7\kappa$ , it follows that  $\chi(M \setminus M(B)) > \kappa$ . By 9.2 applied to the mat  $M \setminus M(A)$  and the sets  $B, Y \setminus B$  (with  $\tau$  replaced by  $\kappa$ ), there exists  $z_2 \in Y \setminus B$  such that

$$\chi((M \setminus M(A)) \cap M(z_2)) > \kappa.$$

By 9.2 again, applied to the mat  $M$  and the sets  $A, \{z_2\}$ , it follows that  $\chi(M(A) \setminus M(z_2)) \leq 2\kappa$ , and consequently  $\chi(M(z_2)) > \tau$ , as required. This proves 9.3.  $\blacksquare$

## 10 Shower completeness

To go further we use a global induction that we explain next. For  $n \geq 2$ , a set of integers is  $n$ -solid if some subset consists of  $n$  consecutive integers. It is 1-solid if it contains two integers that differ by 1 or 3. A key observation is that if a set  $\mathcal{A}$  of integers is  $n$ -solid where  $n > 0$ , then  $\mathcal{A} + \{0, 2\}$  is  $(n + 1)$ -solid. Let us say a shower is  $n$ -complete over a mat  $M$  if its  $M$ -jetset is  $n$ -solid. (For  $n \geq 2$  this agrees with our earlier definition.) Now 8.1 implies that in every candidate, all stable showers with a mat  $M$  of large enough chromatic number are 1-complete over  $M$ ; and as we have seen, to finish the proof of our main theorem 5.1 we only need to show that all stable showers with a mat  $M$  of large enough chromatic number are  $\nu$ -complete over  $M$ .

For  $\sigma > 0$ , let us say an integer  $\zeta \geq 0$  is a *sidekick* for  $\sigma$  if for every candidate  $G$ , and every stable shower  $\mathcal{S}$  in  $G$ ,  $\mathcal{S}$  is  $\sigma$ -complete over  $M$  for every mat  $M$  for  $\mathcal{S}$  with chromatic number more than  $\zeta$ .

Next we need a third inclusion relation for showers, as follows. Let  $\mathcal{S} = (L_0, \dots, L_k, s)$  be a stable shower, and let  $\mathcal{S}' = (L'_0, \dots, L'_{k'}, s')$  be a shower, both in a graph  $G$ ; and let  $P$  be an induced path between  $L_0, L'_0$ . Suppose that

- $s = s'$ ;
- $L'_0, \dots, L'_{k'-1} \subseteq L_0 \cup \dots \cup L_{k-1}$ ;
- $L'_{k'} \subseteq L_k$ ; and
- no vertex of  $P$  belongs to  $L'_1 \cup \dots \cup L'_{k'}$ , and no vertex of  $P$  has a neighbour in this set except the vertex in  $L'_0$ .

In this situation we say that  $\mathcal{S}'$  is *included* in  $\mathcal{S}$ , and  $P$  is a *pipe*. It follows that  $\mathcal{S}'$  is a stable shower, because the subgraph induced on  $L_0 \cup \dots \cup L_{k-1}$  is bipartite. We see that if  $\mathcal{S}'$  is a subshower of  $\mathcal{S}$  then it is included in  $\mathcal{S}$ , via a pipe consisting of a monotone path between the two shower heads.

Let  $\mathcal{S}'$  be included in  $\mathcal{S}$ , with a pipe  $P$ . For every jet  $J'$  of  $\mathcal{S}'$ ,  $J' \cup P$  is a jet of  $\mathcal{S}$ ; and consequently, if the jetsets of the two showers are  $\mathcal{A}, \mathcal{A}'$  respectively then  $\mathcal{A}' + \{|E(P)|\} \subseteq \mathcal{A}$ . Thus if  $\mathcal{S}'$  is  $n$ -complete for some  $n$ , then so is  $\mathcal{S}$ . If  $M, M'$  are mats for  $\mathcal{S}, \mathcal{S}'$  respectively, and  $M' \subseteq M$ , then for every  $M'$ -jet  $J$  of  $\mathcal{S}'$ ,  $J \cup P$  is an  $M$ -jet of  $\mathcal{S}$ ; and so the same relation holds between the  $M$ - and  $M'$ -jetsets of the two showers. Note that the floor of  $\mathcal{S}'$  is a subset of the floor of  $\mathcal{S}$ , but for an individual vertex  $v$ , there may be descendants of  $v$  in  $\mathcal{S}'$  that are not descendants in  $\mathcal{S}$ . (This is not the case for subshowers, and for this reason some results will only work for subshowers.)

Let  $\mathcal{S}'$  be included in  $\mathcal{S}$ . We say a *switch* for  $\mathcal{S}'$  in  $\mathcal{S}$  is a pair  $(P_1, P_2)$  of pipes such that  $|E(P_2)| = |E(P_1)| + 2$ .

**10.1** *Let  $\zeta$  be a sidekick for  $\sigma$ . Let  $\mathcal{S}$  be a stable shower in a candidate  $G$ , and let  $\mathcal{S}$  include a shower  $\mathcal{S}'$ . Let  $M, M'$  be mats for  $\mathcal{S}, \mathcal{S}'$  respectively, with  $M' \subseteq M$ . If  $\mathcal{S}$  is not  $(\sigma + 1)$ -complete over  $M$ , and  $\chi(M') > \zeta$ , then there is no switch for  $\mathcal{S}'$  in  $\mathcal{S}$ .*

**Proof.** Let  $\mathcal{S}, \mathcal{S}'$  have heads  $z_0, z_1$  respectively, and suppose that  $(P_1, P_2)$  is a switch for  $\mathcal{S}'$  in  $\mathcal{S}$ . Let  $\mathcal{A}$  be the  $M$ -jetset of  $\mathcal{S}$ , and let  $\mathcal{A}'$  be the  $M'$ -jetset of  $\mathcal{S}'$ . As we saw above,

$$\mathcal{A}' + \{|E(P_1)|, |E(P_1)| + 2\} \subseteq \mathcal{A}.$$

Since  $\chi(M') > \zeta$  and  $\zeta$  is a sidekick for  $\sigma$ , it follows that  $\mathcal{S}'$  is  $\sigma$ -complete over  $M'$ . Consequently  $\mathcal{A}' + \{|E(P_1)|, |E(P_1)| + 2\}$  is  $(\sigma + 1)$ -complete, and hence so is  $\mathcal{A}$ , a contradiction. This proves 10.1. ■

The first use of 10.1 is the following companion to 9.1:

**10.2** *Let  $\zeta$  be a sidekick for  $\sigma$ . Let  $G$  be a candidate, let  $\mathcal{S} = (L_0, \dots, L_k, s)$  be a stable shower in  $G$  with a mat  $M$ , and suppose that  $\mathcal{S}$  is not  $(\sigma + 1)$ -complete over  $M$ . Let  $z_1 \in L_h$ , where  $0 \leq h < k$ , and let  $A$  be a set of children of  $z_1$ ; and let  $b \notin A$  be another child of  $z_1$ . Let  $P$  be a monotone path between  $b$  and  $L_k \cap M$ , such that some vertex of  $P$  has a child with an ancestor in  $A$ . Let  $P$  have vertices  $p_{h+1} \dots p_k$ , where  $p_{h+1} = b$  and  $p_k \in L_k \cap M$ . If  $i \geq h + 2$  and  $p_i$  has a child  $x$  which has an ancestor in  $A$ , then the set of descendants of  $p_i$  in  $M$  that are not descendants of  $x$  has chromatic number at most  $\zeta$ .*

**Proof.** Let  $P_1$  be a monotone path between  $z_0, z_1$ . Since  $x$  has an ancestor in  $A$ , there is a monotone path  $Q$  between  $x, z_0$  containing a vertex in  $A$ . Consequently  $Q$  contains no descendant of  $b$  except  $x$ ; and since  $i > h + 1$ , the path  $P_2$  formed by the union of  $Q$  and the edge  $x p_i$  is induced. Its length is  $|E(P_1)| + 2$ . Let  $\mathcal{S}'$  be the subshower of  $\mathcal{S}$  between  $p_i$  and  $M \setminus \Theta(x)$ , and let  $M'$  be the intersection of its floor with  $M \setminus \Theta(x)$ . This is a stable shower included in  $\mathcal{S}$ , with a mat  $M'$ , and  $(P_1, P_2)$  is a switch for it. By 10.1,  $\chi(M') \leq \zeta$ . This proves 10.2.  $\blacksquare$

## 11 The shadow of a wand

Let  $\mathcal{S} = (L_0, \dots, L_k, s)$  be a stable shower. A *wand*  $\mathcal{W}$  of length  $t$  in  $\mathcal{S}$  is a sequence  $(W_0, \dots, W_t)$  with the following properties:

- $t \leq k - 2$ ;
- $\emptyset \neq W_i \subseteq L_i$  for  $0 \leq i \leq t$ ;
- every vertex in  $W_i$  is adjacent to every vertex in  $W_{i+1}$  for  $0 \leq i \leq t - 1$ .

We define  $V(\mathcal{W}) = W_0 \cup \dots \cup W_t$ .

Let  $(W_0, \dots, W_t)$  be a wand  $\mathcal{W}$  in  $\mathcal{S}$ . If  $u \in W_i$  for some  $i$ , we say that a neighbour  $v$  of  $u$  is an *up-neighbour* of  $u$  if

- $v \notin V(\mathcal{W})$ ;
- $v \in L_{i-1}$  (and therefore  $i \geq 2$ ); and
- every neighbour of  $v$  in  $V(\mathcal{W})$  belongs to  $W_i$  (and therefore  $i \geq 3$ ).

For  $0 \leq i \leq t - 1$ , let  $T_i$  be the set of all vertices  $v \in L_i$  such that  $v$  is an up-neighbour of some vertex in  $W_{i+1}$ . Let  $T = T_0 \cup \dots \cup T_{t-1}$ . For  $u \in T$ , a *post* with *top*  $u$  is a monotone path between  $u$  and  $L_k$  such that no vertex of this path except  $u$  belongs to or has a neighbour in  $V(\mathcal{W})$ . (A post with top  $u$  therefore provides an induced path between the neighbour ( $s$  say) of  $u$  in  $V(\mathcal{W})$  and  $L_k$  of length two more than a monotone path between  $s$  and  $L_k$ , and we shall exploit this later.) For  $0 \leq i \leq k$ , let  $S_i$  be the set of all vertices  $v \in L_i$  that belong to a post with top in  $T$ . (Thus  $T_i \subseteq S_i \subseteq L_i \setminus V(\mathcal{W})$ , and  $S_0 = \emptyset$ .) If  $M$  is a mat for  $\mathcal{S}$ , we call  $M \cap S_k$  the *shadow* (over  $M$ ) of the wand.

Showers in which no wand shadow has large  $\chi$  are easier to work with than general showers. In this section we prove that their mats have bounded chromatic number. The proof requires several steps. We begin with:

**11.1** Let  $\mathcal{S}$  be a stable shower with mat  $M$  in a candidate  $G$ , such that every wand in  $\mathcal{S}$  has shadow over  $M$  with chromatic number at most  $\tau$ . Let  $z_1 \in U(\mathcal{S})$ , and let  $A, B$  be disjoint sets of children of  $z_1$ . If  $\chi(M(A)) > \kappa$  then  $\chi(M(B) \setminus M(A)) \leq (\nu + 1)(3\nu^2 + 1)(\tau + \kappa) + 2\kappa$ .

**Proof.** Suppose not. Let  $\mathcal{S}_1$  be the maximal subshower of  $\mathcal{S}$  with head  $z_1$  such that every vertex in  $U(\mathcal{S}_1)$  except  $z_1$  has an ancestor in  $A$ . Then  $B \cap U(\mathcal{S}_1) = \emptyset$ . Let  $\mathcal{S}_2$  be the maximal subshower of  $\mathcal{S}$  with head  $z_1$  such that every vertex in  $U(\mathcal{S}_2)$  except  $z_1$  has an ancestor in  $B$  and has no ancestor in  $A$ . For  $i = 1, 2$  let  $V_i = U(\mathcal{S}_i)$ . Thus  $V_1 \cap V_2 = \{z_1\}$ , and no vertex in  $V_2$  has a parent in  $V_1$ . Moreover,  $M(A)$  is a mat for  $\mathcal{S}_1$ , and  $M(B) \setminus M(A)$  is a mat for  $\mathcal{S}_2$ . By 9.2, since  $\chi(M(B) \setminus M(A)) > (\nu + 1)(3\nu^2 + 1)(\tau + \kappa) + 2\kappa$ , there is a monotone path  $R$  of  $G[V_1]$  between  $z_1$  and  $M(A)$  such that, if  $X$  denotes the set of vertices in  $V_2 \setminus \{z_1\}$  with a neighbour in  $V(R) \setminus \{z_1\}$ , then the set of vertices in  $M(B) \setminus M(A)$  with an ancestor in  $X$  has chromatic number more than  $\tau + \kappa$ .

Now no vertex of  $R$  different from  $z_1$  has a child in  $V_2$ , so every vertex in  $X$  has a child in  $V(R)$ . Let  $y$  be the vertex of  $R$  with height two, and let  $R'$  be the subpath of  $R$  between  $z_1, y$ . Let  $X_1$  be the set of vertices in  $X$  with a child in  $R'$ , and let  $X_2$  be the set of vertices in  $X$  with a child in  $R$  with height at most one. Let  $P$  be the union of  $R$  and a monotone path between  $L_0$  and  $z_1$ . The vertices of  $P$  in order form a wand, and every vertex in  $X_1$  is an up-neighbour of a vertex of this wand. Consequently the set of descendants in  $M$  of  $X_1$  is a subset of the shadow of this wand, and so has chromatic number at most  $\tau$ . But every vertex with an ancestor in  $X_2$  is at distance at most three from the penultimate vertex of  $R$ , and in particular the set of descendants in  $M$  of  $X_2$  has chromatic number at most  $\kappa$ . Consequently  $\chi(M(X)) \leq \tau + \kappa$ , a contradiction. This proves 11.1. ■

Let  $\mathcal{S}$  be a stable shower in a candidate  $G$ . A wand  $\mathcal{W} = (W_0, \dots, W_t)$  is said to be  $\xi$ -diagonal if

- every vertex of  $U(\mathcal{S})$  with a child in  $V(\mathcal{W})$  belongs to  $V(\mathcal{W})$ ; and
- for  $0 \leq i \leq t$ , the set of vertices in  $M$  that have an ancestor in  $W_i$  and no ancestor in  $W_{i+1}$  has chromatic number at most  $\xi$  (where  $W_{t+1} = \emptyset$ ).

Next we need some results about showers that admits  $\xi$ -diagonal wands, where  $\xi$  is bounded. Before we do so, let us set up some notation for these things.

If  $\mathcal{S} = (L_0, \dots, L_k, s)$  with mat  $M$ , and  $\mathcal{W}$  is a  $\xi$ -diagonal wand  $(W_0, \dots, W_t)$  in  $\mathcal{S}$ , then for every vertex  $v$  of  $U(\mathcal{S}) \cup M$ , there is a maximum  $i \leq t$  such that  $W_i$  contains an ancestor of  $v$ . We call this number  $i$  the *reach* of  $v$  (with respect to  $\mathcal{W}$ ). Let  $V = U(\mathcal{S})$ , and for  $0 \leq i \leq t$  let  $M_i$  and  $V_i$  be the sets of all vertices in  $M$  and in  $V$  with reach  $i$ , respectively. It follows that no member of  $V_j$  has a child in  $V_i$  if  $i < j$ . A monotone path is *vertical* if for some  $i$ , all its vertices belong to  $M_i \cup V_i$ . Let  $0 \leq h \leq t$ , and let  $P$  be a monotone path between some vertex in  $M_h$  and some vertex in  $W_h$ . It follows that  $P$  is vertical. If  $X \subseteq V \cup M$ , the set of vertices in  $M$  joined to a vertex in  $X$  by a vertical path is denoted by  $X \downarrow M$ .

**11.2** Let  $\zeta$  be a sidekick for  $\sigma$ . Let  $\mathcal{S}$  be a stable shower with mat  $M$  in a candidate  $G$ , such that  $\mathcal{S}$  is not  $(\sigma + 1)$ -complete over  $M$ , and the shadow over  $M$  of every wand in  $\mathcal{S}$  has chromatic number at most  $\tau$ . Let  $\mathcal{W} = (W_0, \dots, W_t)$  be a  $\xi$ -diagonal wand, and, with notation as above, let  $0 \leq h \leq t$ ,

and let  $P$  be a monotone path between  $M$  and  $W_h$ , with no vertex in  $V(\mathcal{W})$  except its vertex in  $W_h$ . Let  $N(P)$  be the set of vertices in  $M \cup (V \setminus V(\mathcal{W}))$  with a neighbour in  $V(P)$ . Then

$$\chi((N(P) \downarrow M) \setminus (M_h \cup M_{h+1})) \leq 2\zeta + 2\xi + \kappa + \tau.$$

**Proof.** Let  $P$  have vertices  $p_h \cdots p_k$  in order, where  $p_i \in L_i$  for  $h \leq i \leq k$ . Thus  $p_h \in W_h$ . For  $h \leq i \leq k$ , let be the set of parents of  $p_{i+1}$  in  $V \setminus V(\mathcal{W})$  (taking  $Y_k = \emptyset$ ). Let  $Z_1$  be the set of vertices in  $V \setminus V(\mathcal{W})$  with height at least three, with a child in  $V(P)$  and with no parent in  $V(P)$ . It follows that  $Z_1 \cap V(P) = \emptyset$ .

$$(1) \chi(Z_1 \downarrow M) \leq \tau.$$

The sequence

$$(W_0, \dots, W_{h-1}, \{p_h\}, \{p_{h+1}\}, \dots, \{p_{k-2}\})$$

is a wand, and every vertex in  $Z_1$  is an up-neighbour of a vertex in this wand, and so every vertex in  $Z_1 \downarrow M$  belongs to the shadow of this wand over  $M$ . Since by hypothesis the shadow of every wand over  $M$  has chromatic number at most  $\tau$ , this proves (1).

Let  $Z_2$  be the set of vertices in  $V \setminus V(\mathcal{W})$  adjacent to one of  $p_{k-2}, p_{k-1}, p_k$ .

$$(2) \chi(Z_2 \downarrow M) \leq \kappa.$$

This is immediate since every vertex in  $Z_1 \downarrow M$  has distance at most four from  $p_{k-2}$ , and  $\rho \geq 4$ .

Let  $Z_3$  be the set of vertices in  $V \setminus V(\mathcal{W})$  with a parent in  $\{p_h, \dots, p_{k-3}\}$ .

$$(3) \chi((Z_3 \downarrow M) \setminus (M_h \cup M_{h+1})) \leq 2\zeta + 2\xi.$$

To show this, we may assume that  $Z_3 \downarrow M \not\subseteq M_h \cup M_{h+1}$ . Thus there exists  $j \leq k-2$  and  $i \geq h$  such that  $j \neq h, h+1$  and some child of  $p_i$  belongs to  $V_j$  (as we see by following down the path  $P$  until it leaves  $V_h \cup V_{h+1}$ ). Since no descendant of  $p_h$  belongs to  $V_{h'}$  for  $h' < h$ , it follows that  $j \geq h+2$ . Every vertex in  $V_j$  is a descendent of a vertex in  $W_j$ , and consequently  $i \geq j \geq h+2$ . We have shown then that there exists  $i \in \{h+2, \dots, k-3\}$  such that  $p_i$  has a child in  $V_j$  for some  $j \geq h+2$ . Choose  $i \geq h+2$  minimum with this property; and let  $j_1 \in \{h+2, \dots, k-2\}$  be maximum such that  $p_i$  has a child in  $V_{j_1}$ . If possible, let  $j_2 \in \{h+2, \dots, k-2\}$  be maximum such that  $p_{i+1}$  has a child in  $V_{j_2}$ , and otherwise let  $j_2 = h$ .

Let  $x$  be a child of  $p_i$  in  $V_{j_1}$ . Let  $\mathcal{S}_1$  be the maximal subshower of  $\mathcal{S}$  with head  $p_i$  such that no child of  $x$  belongs to  $U(\mathcal{S}_1)$ , and let  $M^1$  be the set of vertices  $v \in M$  such that there is a monotone path of  $\mathcal{S}$  between  $v, p_i$  containing no child of  $x$ . Thus,  $M^1$  is a mat for  $\mathcal{S}_1$ . For  $0 \leq a \leq k-2$ , let  $c_a \in W_a$ , where  $c_h = p_h$ . Let  $Q$  be a vertical path between  $x$  and  $c_j$ ; then  $p_i$  has no neighbour in  $V(Q)$ . Consequently  $c_0 \cdots c_h - p_{h+1} \cdots - p_i$  and  $c_0 - c_j - Q - x - p_i$  are both induced paths, and so the pair form a switch for  $\mathcal{S}_1$ . From 10.1, it follows that  $\chi(M^1) \leq \zeta$ .

If  $j_2 > h$  let  $x_2$  be a child of  $p_{i+1}$  in  $V_{j_2}$ , and let  $M^2$  be the set of vertices  $v \in M$  such that there is a monotone path of  $\mathcal{S}$  between  $v, p_{i+1}$  containing no child of  $x_2$ ; then similarly,  $\chi(M^2) \leq \zeta$ . (If  $j_2 = h$  let  $M^2 = \emptyset$ .)

Let  $M^3$  be the set of all  $v \in Z_3 \downarrow M$  such that  $v \notin M^1 \cup M^2$ . Let  $v \in M^3$ ; since  $v \in Z_3 \downarrow M$  there is a vertical path  $R$  between  $v$  and a child of  $p_{i'}$  for some  $i'$  with  $h \leq i' \leq k-3$ . Let  $v \in M_{j'}$ ; then  $R$  can be extended to a vertical path between  $v$  and  $W_{j'}$ . Since  $v \notin M^1$ , this vertical path contains a child of  $x$ . Every child of  $x$  belongs to  $V_{j_1} \cup \dots \cup V_{k-2}$ , and so  $j' \geq j_1$ . Since there is a child of  $p_{i'}$  in  $R$  and hence in  $V_{j'}$ , it follows from the choice of  $i$  that either  $i' \geq i$  or  $j' = h+1$ . If  $i' \geq i+2$  there is a monotone path between  $v, p_i$  containing no child of  $x$ , a contradiction; so  $i' \leq i+1$ . Hence either  $i' = i$ , or  $i' = i+1$ , or  $j' = h+1$ . If  $i' = i$ , then the choice of  $j_1$  implies that  $j' \leq j_1$ ; and since a child of  $x$  belongs to  $V_{j'}$ , it follows that  $j' = j_1$ . Similarly, if  $i' = i+1$ , then the choice of  $j_2$  implies that  $j' \leq j_2$ ; and since a child of  $x_2$  belongs to  $V_{j'}$ , it follows that  $j' = j_2$ . We have shown then that  $j'$  is one of  $j_1, j_2, h+1$ . Consequently  $M^3 \subseteq M_{h+1} \cup M_{j_1} \cup M_{j_2}$ .

But  $Z_3 \downarrow M \subseteq M^1 \cup M^2 \cup M^3$ , and  $\chi(M_{j_1} \cup M_{j_2}) \leq 2\xi$ , so  $\chi((Z_3 \downarrow M) \setminus (M_h \cup M_{h+1})) \leq 2\xi + 2\zeta$ . This proves (3).

From (1), (2) and (3), since  $N(P) = Z_1 \cup Z_2 \cup Z_3$ , the result follows. This proves 11.2.  $\blacksquare$

**11.3** Let  $\zeta$  be a sidekick for  $\sigma$ . Let  $\mathcal{S}$  be a stable shower with mat  $M$  in a candidate  $G$ , such that  $\mathcal{S}$  is not  $(\sigma+1)$ -complete over  $M$ , and the shadow over  $M$  of every wand in  $\mathcal{S}$  has chromatic number at most  $\tau$ . Let  $\mathcal{W}$  be a  $\xi$ -diagonal wand. In the usual notation, let  $h < j \leq t$ , and let  $A \subseteq \bigcup_{h < i < j} M_i$  such that  $G[A]$  is connected and  $\chi(A) > 2\zeta + 5\xi + 2\kappa + \tau$ . Let  $c_h \in W_h$ , and  $c_j \in W_j$ . Then there is a set  $\mathcal{A}$  of integers, and for each  $a \in \mathcal{A}$  there is an induced path  $J_a$  of  $G$  between  $c_h, c_j$ , with the following properties:

- $\mathcal{A}$  has cardinality at most  $\nu + 1$ , and includes a dense set of cardinality  $\nu$ , and contains two integers  $x, y$  with  $y - x \in \{1, 3\}$ ;
- $|E(J_a)| = a$  for each  $a \in \mathcal{A}$ ;
- for each  $a \in \mathcal{A}$ ,  $V(J_a) \setminus \{c_h, c_j\} \subseteq V_{h+1} \cup \dots \cup V_{j-1} \cup A$ ;
- for each  $a \in \mathcal{A}$ , there is a set of at most  $3\nu^2 + 2$  monotone paths between  $A$  and  $c_h$ , such that every vertex of  $V(J_a) \setminus (A \cup V(\mathcal{W}))$  belongs to one of these paths.

**Proof.** No vertex in  $A$  has an ancestor in  $W_j$ ; choose  $i < j$  maximum such that some  $c_i \in W_i$  has a descendant in  $A$ . Let  $Q$  be a monotone path between  $c_i$  and  $A$ . By 11.2, there exists  $A' \subseteq A$  such that  $G[A']$  is connected,

$$\chi(A') \geq \chi(A) - (2\zeta + 5\xi + \kappa + \tau) \geq \kappa,$$

and no vertical path meets both  $N(Q) \cup W_i \cup W_{i-1}$  and  $A'$ . Let  $\mathcal{S}'$  be the maximal subshower of  $\mathcal{S}$  with head  $c_h$  such that  $U(\mathcal{S}')$  contains no vertex of  $N(Q) \cup W_i \cup W_{i-1}$ ; then  $A'$  is a mat for  $\mathcal{S}'$ . Let  $\mathcal{S}'$  be  $L'_j, \dots, L'_{k-1}, L_k, s$  say. Let  $L'_k$  be the union of  $A, V(Q)$ , and  $W_{i+1} \cup W_{i+2} \cup \dots \cup W_j$ ; then  $G[L'_k]$  is connected, and every vertex of  $U(\mathcal{S}')$  with a neighbour in  $L'_k$  belongs to  $L'_{k-1}$ . Thus  $L'_j, \dots, L'_{k-1}, L'_k, c_j$  is a shower, with mat  $A'$ ; and the result follows from 8.1 and 8.4. (Note: 8.4 gives us  $3\nu^2 + 1$  monotone paths containing all the vertices of  $J_a$  not in  $L'_k$ ; but we also need to cover the vertices of  $J_a$  in  $L'_k \setminus L_k$ . One more monotone path will do this, namely  $Q$ .) This proves 11.3.  $\blacksquare$

**11.4** Let  $\zeta$  be a sidekick for  $\sigma$ . Let  $\mathcal{S}$  be a stable shower with mat  $M$  in a candidate  $G$ , such that  $\mathcal{S}$  is not  $(\sigma + 1)$ -complete over  $M$ , and the shadow over  $M$  of every wand in  $\mathcal{S}$  has chromatic number at most  $\tau$ . Let  $\mathcal{W}$  be a  $\xi$ -diagonal wand. With the usual notation, let  $j_0 < j_1 < j_2 \leq t$ , and suppose that  $u_1 \in M_{j_0}$  and  $u_2 \in M_{j_2}$  are adjacent. Let  $M^1 \subseteq \bigcup_{j_0 < j < j_1} M_j$  and  $M^2 \subseteq \bigcup_{j_1 < j < j_2} M_j$ . If  $\chi(M^1) > 6\zeta + 13\xi + 4\kappa + 3\tau$  and

$$\chi(M^2) > (2\zeta + 4\xi + \kappa + \tau)(3 + (\nu + 1)(3\nu^2 + 2)) + \xi + \kappa$$

then there is an edge between  $M^1, M^2$ .

**Proof.** Let  $P_1$  be a vertical path between  $u_1, W_{j_0}$ , and let  $P_2$  be a vertical path between  $u_2, W_{j_2}$ . Let  $c_{j_0}$  be the end of  $P_1$  in  $W_{j_0}$ , and let  $c_{j_2}$  be the end of  $P_2$  in  $W_{j_2}$ . Since  $u_1, u_2$  are adjacent, there is an induced path  $P$  between  $c_{j_0}, c_{j_2}$  with  $V(P) \subseteq V(P_1 \cup P_2)$ . For  $i = 1, 2$ , let  $N(P_i)$  the set of vertices in  $M \cup (V \setminus V(\mathcal{W}))$  with a neighbour in  $V(P_i)$ . By 11.2, for  $i = 1, 2$ ,  $\chi(N(P_i) \downarrow M) \leq 2\zeta + 4\xi + \kappa + \tau$ , and so there exists  $A_1 \subseteq M^1$  with  $\chi(A_1) > 2\zeta + 5\xi + 2\kappa + \tau$ , such that  $G[A_1]$  is connected and no vertex in  $A_1$  belongs to a vertical path that intersects  $N(P_1) \cup N(P_2)$ . Choose  $c_{j_1} \in W_{j_1}$ . By 11.3,

(1) There is a set  $\mathcal{A}$  of integers, and for each  $a \in \mathcal{A}$  there is an induced path  $J_a$  of  $G$  between  $c_{j_0}, c_{j_1}$ , with the following properties:

- $\mathcal{A}$  has cardinality at most  $\nu + 1$ , and includes a dense set of cardinality  $\nu$ , and contains two integers  $x, y$  with  $y - x \in \{1, 3\}$ ;
- $|E(J_a)| = a$  for each  $a \in \mathcal{A}$ ;
- for each  $a \in \mathcal{A}$ ,  $V(J_a) \subseteq V_{j_0+1} \cup \dots \cup V_{j_1-1} \cup A_1 \cup \{c_{j_0}, c_{j_1}\}$ ;
- for each  $a \in \mathcal{A}$ , there is a set of at most  $3\nu^2 + 2$  monotone paths between  $A_1$  and  $c_{j_0}$ , such that every vertex of  $V(J_a) \setminus (A_1 \cup V(\mathcal{W}))$  belongs to one of these paths.

Now suppose that there are no edges between  $M^1, M^2$ . By  $(\nu + 1)(3\nu^2 + 2) + 2$  applications of 11.3, there exists  $A_2 \subseteq M_2$  with the following properties:

- $G[A_2]$  is connected, and no vertex in  $A_2$  belongs to a vertical path that intersects  $N(P_1) \cup N(P_2)$ ;
- for each  $a \in \mathcal{A}$ , no vertex in  $A_2$  belongs to a vertical path that contains a vertex in  $V(J_a)$  or a neighbour of such a vertex (here we use that there is no edge between  $A_1$  and  $M^2$ )
- $\chi(A_2) > 2\zeta + 5\xi + 2\kappa + \tau$ .

We apply 11.3 to  $A_2$ , and thereby obtain a set of paths joining  $c_{j_1}$  and  $c_{j_2}$ . But for each of these paths, say  $J$ , and each  $a \in \mathcal{A}$ , the union  $J \cup J_a$  is an induced path between  $c_{j_0}, c_{j_2}$ ; and it can be combined with the induced path  $P$  to form a hole. It follows as usual that  $G$  contains a hole  $\nu$ -interval, a contradiction. This proves 11.4. ■

**11.5** Let  $\zeta$  be a sidekick for  $\sigma$ . Let  $\mathcal{S}$  be a stable shower with mat  $M$  in a candidate  $G$ , such that  $\mathcal{S}$  is not  $(\sigma + 1)$ -complete over  $M$ , and the shadow over  $M$  of every wand in  $\mathcal{S}$  has chromatic number at most  $\tau$ . Let  $\mathcal{W}$  be a  $\xi$ -diagonal wand. In the usual notation, let  $j_1 < j_2 \leq t$ , and suppose that  $u_1 \in M_{j_1}$  and  $u_2 \in M_{j_2}$  are adjacent. Let  $M^1 \subseteq \bigcup_{j_1 < j < j_2} M_j$  and  $M^2 \subseteq \bigcup_{j_2 < j \leq t} M_j$ . If

$$\chi(M^1) > 2(2\zeta + 4\xi + \kappa + \tau) + \kappa$$

and

$$\chi(M^2) > 2(2\zeta + 4\xi + \kappa + \tau) + 2\kappa + (\nu + 1)(3\nu^2 + 1)\tau$$

then there exist  $A_1 \subseteq M^1$  and  $A_2 \subseteq M^2$  such that  $\chi(A_i) \geq \chi(M^i) - 2(2\zeta + 4\xi + \kappa + \tau)$  for  $i = 1, 2$ , and such that there is no edge between  $A_1, A_2$ .

**Proof.** For  $i = 1, 2$ , let  $P_i$  be a vertical path between  $u_i$  and some  $c_{j_i} \in W_{j_i}$ . Let  $P$  be an induced path between  $c_{j_1}, c_{j_2}$  with  $V(P) \subseteq V(P_1 \cup P_2)$ . For  $i = 1, 2$ , let  $N(P_i)$  the set of vertices in  $M \cup (V \setminus V(\mathcal{W}))$  with a neighbour in  $V(P_i)$ .

Let  $B$  be the set of all vertices that belong to a vertical path  $R$  between  $M^1 \cup M^2$  and  $V(\mathcal{W})$  such that no vertex of  $R$  belongs to  $N(P_1) \cup N(P_2)$ . Then there is a subshower  $\mathcal{S}'$  of  $\mathcal{S}$  with head  $c_{j_1}$  such that  $U(\mathcal{S}') \setminus V(\mathcal{W}) = B \setminus L_k$ . (Some vertices of  $\mathcal{S}'$  belong to  $V(\mathcal{W})$ .) Let

$$\mathcal{S}' = (L'_{j_1}, L'_{j_1+1}, \dots, L'_{k-1}, L_k, s).$$

By 11.2, for  $i = 1, 2$ ,  $\chi(N(P_i) \downarrow M) \leq 2\zeta + 4\xi + \kappa + \tau$ , and so  $\chi(B \cap M_1) > \chi(M_1) - 2(2\zeta + 4\xi + \kappa + \tau)$ . Choose  $A_1 \subseteq B \cap M_1$ , such that  $G[A_1]$  is connected and  $\chi(A_1) = \chi(B \cap M_1)$ . Similarly, we may choose  $A_2 \subseteq B \cap M_2$  such that  $G[A_2]$  is connected and  $\chi(A_2) > \chi(M_2) - 2(2\zeta + 4\xi + \kappa + \tau)$ .

Suppose that there is an edge between  $A_1, A_2$ . Then  $G[A_1 \cup A_2]$  is connected, and so

$$(L'_{j_1}, L'_{j_1+1}, \dots, L'_{k-1}, A_1 \cup A_2, s_0)$$

is a shower  $\mathcal{S}_0$  say (where  $s_0 \in A_1 \cup A_2$  is arbitrary). Let  $\mathcal{S}_1$  be the maximal subshower of  $\mathcal{S}_0$  with head  $c_{j_1}$  such that every vertex in  $U(\mathcal{S}_1) \setminus V(\mathcal{W})$  belongs to a vertical path of  $\mathcal{S}$  with one end in  $M^1$ , and let  $\mathcal{S}_2$  be the maximal subshower of  $\mathcal{S}_0$  with head  $c_{j_2}$  such that every vertex in  $U(\mathcal{S}_2) \setminus V(\mathcal{W})$  belongs to a vertical path of  $\mathcal{S}$  with one end in  $M^2$ . It follows that  $A_i$  is a mat for  $\mathcal{S}_i$  for  $i = 1, 2$ . Now there is no monotone path  $R$  of  $G[V_2]$  between  $c_{j_2}$  and  $M_2$  such that  $\chi(M_1(X(R))) > \tau$ , where  $X(R)$  denotes the set of vertices in  $V_1$  with a child in  $V(R)$ ; and so by 9.1 (with  $M_1, M_2$  replaced by  $A_2, A_1$ ) since  $\chi(A_2) > \kappa$ , and  $\chi(A_1) > 2\kappa + (\nu + 1)(3\nu^2 + 1)\tau$ , there are  $\nu$  induced paths  $Q_0, \dots, Q_{\nu-1}$  of  $G[V_1 \cup V_2 \cup L_k]$  between  $c_{j_1}, c_{j_2}$ , such that  $|E(Q_i)| = |E(Q_0)| + i$  for  $0 \leq i < \nu$ . But each of these paths forms a hole when combined with  $P$ ; and so  $G$  contains a hole  $\nu$ -interval, which is impossible.

It follows that there is no edge between  $A_1, A_2$ . This proves 11.5.  $\blacksquare$

We need the following lemma.

**11.6** Let  $G$  be a graph with chromatic number more than  $4N$ , and let  $M_1, \dots, M_k$  be a partition of  $V(G)$  such that  $\chi(G[M_i]) \leq N$  for  $1 \leq i \leq k$ . Then there exist  $a < b < c < d < e \leq k$  such that there is an edge of  $G$  between  $M_a$  and  $M_c$ , and an edge between  $M_a$  and  $M_e$ .



**Proof.** Let  $J$  be the graph with vertex set  $\{1, \dots, k\}$  in which  $i, j$  are adjacent if there is an edge of  $G$  between  $M_i$  and  $M_j$ . If  $J$  is 4-colourable, then  $\chi(G) \leq 4N$ , a contradiction. So  $J$  is not 4-colourable, and consequently there exists  $a \in \{1, \dots, k\}$  such that  $a$  is adjacent in  $J$  to at least four of  $a+1, \dots, k$ . Let  $a, b, c, d$  be the four neighbours, in order; then the theorem holds. This proves 11.6.  $\blacksquare$

**11.7** Let  $\zeta$  be a sidekick for  $\sigma$ . Let  $\mathcal{S}$  be a stable shower with mat  $M$  in a candidate  $G$ , such that  $\mathcal{S}$  is not  $(\sigma+1)$ -complete over  $M$ , and the shadow over  $M$  of every wand in  $\mathcal{S}$  has chromatic number at most  $\tau$ . Let  $\mathcal{W}$  be a  $\xi$ -diagonal wand. Let

$$\eta = (2\zeta + 4\xi + \kappa + \tau)(5 + (\nu + 1)(3\nu^2 + 2)) + \xi + \kappa.$$

Then  $\chi(M) \leq 4(\eta + \xi) + \eta$ .

**Proof.** Suppose that  $\chi(M) > 4(\eta + \xi) + \eta$ . Let  $j_0 = 0$ , and define  $j_1, j_2, \dots, j_t$  and  $M^1, \dots, M^{t-1}$  inductively as follows. Having defined  $j_0, \dots, j_i$  and  $M^0, \dots, M^{i-1}$ , if  $\chi(\bigcup_{j_i < j \leq 2k-2} M_j) < \eta$  the sequence terminates; define  $t = i$ . Otherwise choose  $j_{i+1} \leq 2k-2$  minimum such that  $\chi(\bigcup_{j_i < j \leq j_{i+1}} M_j) \geq \eta$ . Let  $M^i = \bigcup_{j_i < j \leq j_{i+1}} M_j$ .

This completes the inductive definition. We see that the sets  $M^1, \dots, M_{t-1}$  are disjoint, and their union has chromatic number at least  $\chi(M) - \eta$ ; and each  $M_i$  has chromatic number at least  $\eta$ , and at most  $\eta + \xi$  (from the minimality of  $j_{i+1}$ ). Since  $\chi(M) > 4(\eta + \xi) + \eta$ , it follows from 11.6 that there exist  $a < b < c < d < e \leq t$  such that there is an edge of  $G$  between  $M^a$  and  $M^c$ , and an edge between  $M^a$  and  $M^e$ . Now

$$\eta \geq 2(2\zeta + 4\xi + \kappa + \tau) + 2\kappa + (\nu + 1)(3\nu^2 + 1)\tau$$

so by 11.5 applied to  $M^b, M^d$  and the edge between  $M^a, M^c$ , there exist  $A_1 \subseteq M^b$  and  $A_2 \subseteq M^d$  such that  $\chi(A_i) \geq \eta - 2(2\zeta + 4\xi + \kappa + \tau)$  for  $i = 1, 2$ , and there is no edge between  $A_1, A_2$ . But this contradicts 11.4 applied to  $A_1, A_2$  and the edge between  $M^a, M^e$ . This completes the proof of 11.7.  $\blacksquare$

Now we can prove the objective of this section, the following.

**11.8** Let  $\zeta$  be a sidekick for  $\sigma$ . Let  $N = (3\nu^2 + 2)(\nu + 1) + 5$ . Let  $\tau \geq 0$ , and let  $\mathcal{S}$  be a stable shower with mat  $M$  in a candidate  $G$ , such that  $\mathcal{S}$  is not  $(\sigma+1)$ -complete over  $M$ , and the shadow over  $M$  of every wand in  $\mathcal{S}$  has chromatic number at most  $\tau$ . Then  $\chi(M) \leq 40N\zeta + 80N^2(\tau + \kappa)$ .

**Proof.** Let

$$\xi = (\nu + 1)(3\nu^2 + 1)(\tau + \kappa) + 2\kappa,$$

and let

$$\eta = (2\zeta + 4\xi + \kappa + \tau)(5 + (\nu + 1)(3\nu^2 + 2)) + \xi + \kappa.$$

Let  $z_0 \in L_0$ , and recursively, having defined  $z_i$ , let  $z_{i+1}$  be a child of  $z_i$  chosen such that  $\chi(M(z_{i+1})) > \kappa$  if there is such a child; otherwise the definition terminates, when  $i = t$  say. Thus  $M$  is the union of the sets  $M(z_i)$  for  $0 \leq i \leq t$ .

(1) For  $0 \leq i < t$ ,  $\chi(M(z_i) \setminus M(z_{i+1})) \leq \xi$ , and  $\chi(M(z_t)) \leq \xi$ .

For  $0 \leq i < t$ , since  $\chi(M(z_{i+1})) > \kappa$ , 11.1 implies that

$$\chi(M(z_i) \setminus M(z_{i+1})) \leq (\nu + 1)(3\nu^2 + 1)(\tau + \kappa) + 2\kappa \leq \xi.$$

Every child  $z$  of  $z_t$  satisfies  $\chi(M(z)) \leq \kappa$ , and so by 9.3,

$$\chi(M(z_t)) \leq ((\nu + 1)(3\nu^2 + 1) + 8)\kappa \leq \xi.$$

This proves (1).

For each vertex  $v \in M$ , choose a monotone path  $R_v$  between  $v$  and some vertex  $x_v$ , such that  $x_v$  has a neighbour in  $\{z_0, \dots, z_t\}$ , with minimum length. Thus no vertex of  $R_v$  except  $x_v$  has a neighbour in  $\{z_0, \dots, z_t\}$ . Now  $x_v$  might have a parent in  $\{z_0, \dots, z_t\}$ , or a child, or both. Since  $(\{z_0\}, \dots, \{z_t\})$  is a wand, it follows that the set of all  $v$  such that  $x_v$  has a child but no parent in  $\{z_0, \dots, z_t\}$  has chromatic number at most  $\tau$ . The set of all  $v$  such that  $x_v$  has a parent but no child in  $\{z_0, \dots, z_t\}$  has chromatic number at most  $4(\eta + \xi) + \eta$ , by 11.5, applied to the subshower induced on the union of the vertex sets of the corresponding paths  $R_v$ , together with  $\{z_0, \dots, z_t\}$  (because in this subshower, the wand is  $\xi$ -diagonal).

It remains then to bound the chromatic number of the set  $M'$  of  $v \in M$  such that  $x_v$  has both a parent and a child in  $\{z_0, \dots, z_t\}$ . Let  $\mathcal{S}'$  be the subshower induced on the union of the vertex sets of the corresponding paths  $R_v$ , together with  $\{z_0, \dots, z_t\}$ . For  $1 \leq i \leq t-1$ , let  $D_i$  be the set of all vertices of  $U(\mathcal{S}')$  that are adjacent to both  $z_{i+1}, z_{i-1}$ , and let  $D_0 = \{z_0\}$  and  $D_t = \{z_t\}$ . (Note that  $z_t$  is the only child of  $z_{t-1}$  in  $U(\mathcal{S}')$ ). Thus  $z_i \in D_i$  for all  $i$ . For  $c = 0, 1, 2$ , let  $\mathcal{W}_c$  be the sequence  $X_0, \dots, X_t$ , where  $X_i = D_i$  if  $i = c$  modulo 3, and otherwise  $X_i = \{z_i\}$ .

Thus each  $\mathcal{W}_c$  is a wand, and for each  $v \in M'$ ,  $x_v \in V(\mathcal{W}_c)$  for some  $c \in \{0, 1, 2\}$ . For  $c = 0, 1, 2$ , let  $M'_c$  be the set of  $v \in M'$  such that  $x_v \in D_i$  for some  $i \in \{0, \dots, t\}$  congruent to  $c$  modulo 3. Let  $c \in \{0, 1, 2\}$ , and let  $v \in M'_c$ . Now no vertex of  $R_v \setminus \{x_v\}$  has a parent in  $V(\mathcal{W}_c)$ , from the minimality of the length of  $R_v$ , except for the child of  $x_v$  in  $R_v$ ; and the latter has no child in  $V(\mathcal{W}_c)$  since it has no neighbour in  $\{z_0, \dots, z_t\}$ . Consequently, if some vertex in  $R_v \setminus \{x_v\}$  has a child in  $V(\mathcal{W}_c)$ , then  $v$  belongs to the shadow of the wand  $\mathcal{W}_c$  in  $\mathcal{S}'$ ; and so the set of all such  $v$  has chromatic number at most  $\tau$ .

Finally, the set of  $v \in M'_c$  such that no vertex in  $R_v \setminus \{x_v\}$  has a child in  $V(\mathcal{W}_c)$ , has chromatic number at most  $4(\eta + \xi) + \eta$ , by 11.5. In total then, we have shown that

$$\chi(M) \leq \tau + (4(\eta + \xi) + \eta) + 3(\tau + (4(\eta + \xi) + \eta)) = 4\tau + 20\eta + 16\xi.$$

After some arithmetic, which we omit, this proves 11.8. ■

## 12 Raising a wand

Now we turn to general showers, in which a wand shadow may have large chromatic number. We will prove that, if there is such a wand, then we can use it to construct a new shower, still with large  $\chi$ , in which no wand shadow has large chromatic number. We begin with:

**12.1** Let  $\mathcal{S} = (L_0, \dots, L_k, s)$  be a stable shower in a candidate  $G$ , and let  $(W_0, \dots, W_t)$  be a wand  $\mathcal{W}$  in  $\mathcal{S}$ . Let  $v$  be a vertex of some post, and let  $v \in L_i$  say. Then there are two induced paths  $P_1, P_2$  of  $G$  between  $v$  and  $L_0$ , such that  $|E(P_2)| = |E(P_1)| + 2$ , and for  $j > i$  every vertex in  $L_j$  that belongs to or has a neighbour in either of these paths belongs to  $W_{i+1} \cup W_{i+2} \cup \{v\}$ .

**Proof.** Let  $P$  be a post containing  $v$ , with top  $t \in T_h$  say; thus  $h \leq i$ . Let  $P_0$  be the subpath of  $P$  between  $v, t$ . Let  $u \in W_{h+1}$  be adjacent to  $t$ . Let  $P_1$  be the union of  $P_0$  and a monotone path between  $t$  and  $L_0$ .  $P_2$  be the union of  $P_0$ , the edge  $tu$ , and a path between  $u$  and  $W_0$  with one vertex in each of  $W_0, \dots, W_{h+1}$ . This proves 12.1. ■

**12.2** Let  $\zeta$  be a sidekick for  $\sigma$ . Let  $\mathcal{S} = (L_0, \dots, L_k, s)$  be a stable shower in a candidate  $G$ , with mat  $M$ , such that  $\mathcal{S}$  is not  $(\sigma + 1)$ -complete over  $M$ . Let  $(W_0, \dots, W_t)$  be a wand  $\mathcal{W}$  in  $\mathcal{S}$ . Let  $0 \leq i \leq t - 1$ , and let  $T_i$  be the set of up-neighbours of vertices in  $W_{i+1}$ . Let  $M'$  be the set of all  $v \in M$  that belong to a post with top in  $T_i$ . Then

$$\chi(M') \leq \zeta + 2((\nu + 1)(3\nu^2 + 1) + 7)\kappa.$$

**Proof.** For  $X \subseteq T_i$ , and  $j \in \{i, \dots, k\}$ , let  $L_j(X)$  be the set of all vertices in  $L_j$  that belong to a post with top in  $X$ . Then

$$(W_0, W_1, \dots, W_i, X, L_i(X), L_{i+1}(X), \dots, L_{k-1}(X), L_k, s)$$

is a stable shower  $\mathcal{S}(X)$  included in  $\mathcal{S}$  (with a one-vertex pipe); although it is not a subshower. Also  $M' = M \cap L_k(T_i)$ . By 9.3 applied to  $\mathcal{S}(T_i)$  (taking  $z_1 \in W_i$  and  $Y = W_{i+1}$ ) there exists  $u \in W_{i+1}$  such that

$$\chi(M \cap L_k(X)) \geq \chi(M') - ((\nu + 1)(3\nu^2 + 1) + 7)\kappa,$$

where  $X$  is the set of up-neighbours of  $u$ . By 9.3 applied to  $\mathcal{S}(T(X))$  (taking  $z_1 = u$ , and  $Y = X$ ) there exists  $x \in X$  such that

$$\chi(M \cap L_k(x)) \geq \chi(M \cap L_k(X)) - ((\nu + 1)(3\nu^2 + 1) + 7)\kappa;$$

and so

$$\chi(M \cap L_k(x)) \geq \chi(M') - 2((\nu + 1)(3\nu^2 + 1) + 7)\kappa.$$

Now

$$(x, L_i(X), L_{i+1}(X), \dots, L_{k-1}(X), L_k, s)$$

is also a shower included in  $\mathcal{S}$  (with pipe a monotone path between  $x$  and  $L_0$ ), and  $M \cap L_k(x)$  is a mat for it. But there is a switch for  $\mathcal{S}(\{x\})$  in  $\mathcal{S}$ , by 12.1. From 10.1 it follows that  $\chi(M \cap L_k(x)) \leq \zeta$ . We deduce that

$$\chi(M \cap L_k(X)) \leq \zeta + 2((\nu + 1)(3\nu^2 + 1) + 7)\kappa.$$

This proves 12.2. ■

Let  $T_0, \dots, T_{t-1}, T$  be as before. For  $0 \leq i \leq k$ , let  $S_i$  be the set of all vertices  $v \in L_i$  that belong to a post with top in  $T$ . (Thus  $T_i \subseteq S_i \subseteq L_i \setminus V(\mathcal{W})$ , and  $S_0 = \emptyset$ .) If  $M$  is a mat for  $\mathcal{S}$ , it follows (since  $t \leq k-2$ ) that

$$(W_0, W_1, W_2, W_3 \cup S_1, W_4 \cup S_2, \dots, W_t \cup S_{t-2}, S_{t-1}, \dots, S_{k-2}, S_{k-1}, L_k, s)$$

is a stable shower  $\mathcal{S}'$  included in  $\mathcal{S}$ ; and we say that  $\mathcal{S}'$  is obtained from  $\mathcal{S}$  by *raising* the wand. Moreover, the shadow  $M \cap S_k$  is a mat for  $\mathcal{S}'$ .

When we raise a wand, the new shower  $\mathcal{S}'$  is included in  $\mathcal{S}$ , but it is not a subshower, and we must be cautious with concepts such as “child”, “descendant”, because they depend on which shower we are using. For clarity we temporarily replace them by expressions like “ $\mathcal{S}$ -child”.

**12.3** *Let  $\mathcal{S} = (L_0, \dots, L_k, s)$  be a stable shower in a candidate  $G$ , and let  $(W_0, \dots, W_t)$  be a wand in  $\mathcal{S}$ . Let  $\mathcal{S}'$  be obtained from  $\mathcal{S}$  by raising the wand. Then for  $0 \leq i \leq t$ , if  $v \in W_i$  and  $v$  is an  $\mathcal{S}'$ -child of  $u$  then  $i > 0$  and  $u \in W_{i-1}$ .*

**Proof.** In the notation given before, since  $v \in W_i$  and  $v$  is an  $\mathcal{S}'$ -child of  $u$ , it follows that  $i > 0$  and  $u \in W_{i-1} \cup S_{i-3}$ , where  $S_{-1}, S_{-2} = \emptyset$ . But  $S_{i-3} \subseteq L_{i-3}$  and  $v \in W_i \subseteq L_i$ , so  $u \notin S_{i-3}$ , and hence  $u \in W_{i-1}$ . This proves 12.3.  $\blacksquare$

**12.4** *Let  $\zeta$  be a sidekick for  $\sigma$ . Let  $\mathcal{S} = (L_0, \dots, L_k, s)$  be a stable shower in a candidate  $G$ , with mat  $M$ , such that  $\mathcal{S}$  is not  $(\sigma + 1)$ -complete over  $M$ . Suppose that  $\mathcal{S}$  is obtained from some stable shower  $\mathcal{S}_0$  in  $G$  with mat  $M_0$  by raising some wand, and  $M$  is the shadow over  $M_0$  of this wand. Let  $\mathcal{W}$  be a wand in  $\mathcal{S}$ . Then the shadow  $M'$  of  $\mathcal{W}$  over  $M$  has chromatic number at most*

$$3\zeta + 6((\nu + 1)(3\nu^2 + 1) + 7)\kappa.$$

**Proof.** Let  $\mathcal{W} = (W_0, \dots, W_t)$ , and for  $0 \leq i \leq t-1$ , let  $T_i$  be the set of up-neighbours of vertices in  $W_{i+1}$  and let  $T = T_0 \cup \dots \cup T_{t-1}$ . Thus  $M'$  is the set of all  $v \in M$  that belong to a post with top in  $T$ . Choose  $h$  minimum such that  $T_h \neq \emptyset$ . Let  $M_1, M_2$  be the sets of vertices in  $M$  that belong to posts with top in  $T_h \cup T_{h+1}$  and with top in  $T \setminus (T_h \cup T_{h+1})$  respectively. In view of 12.2 it suffices to bound  $\chi(M_2)$ . For  $j = h+2, \dots, k$  let  $S_j$  be the set of vertices in  $L_j$  that belong to a post with top in  $T \setminus (T_h \cup T_{h+1})$ . Thus every vertex of every such post belongs to  $S_j$  for some  $j$ . Choose  $u \in W_{h+1}$  with a neighbour  $t \in T_h$ . Consequently

$$(\{u\}, W_{h+2}, W_{h+3}, W_{h+4} \cup S_{h+2}, W_{h+5} \cup S_{h+3}, \dots, W_t \cup S_{t-2}, S_{t-1}, \dots, S_{k-1}, L_k, s)$$

is a shower  $\mathcal{S}'$ , and  $M_2$  is a mat for it. Every vertex of  $U(\mathcal{S}')$  belongs to  $L_j$  for some  $j \geq h+2$ , except  $u$ . We claim there is a switch for this shower; but in  $\mathcal{S}_0$ , not in  $\mathcal{S}$ .

Let  $\mathcal{S}_0$  be  $(J_0, \dots, J_{k-3}, L_k, s)$ . Now  $\mathcal{S}$  is obtained from  $\mathcal{S}_0$  by raising some wand  $\mathcal{D}$  say, where  $M$  is the shadow of  $\mathcal{D}$  on some mat  $M_0$  for  $\mathcal{S}_0$ . Let  $\mathcal{D}$  be  $D_0, \dots, D_r$ , and set  $D_i = \emptyset$  for  $i > r$ ; then for  $0 \leq i \leq t$ ,  $L_i \subseteq D_i \cup (J_{i-2} \setminus V(\mathcal{D}))$  (where  $J_{-1}, J_{-2} = \emptyset$ ). Suppose that  $u \in V(\mathcal{D})$ ; then since  $u \in L_{h+1}$ , it follows that  $u \in D_{h+1}$ . Every vertex of  $W_{h-1}$  has distance two from  $u$ , and so  $W_{h-1} \cap J_{h-3} = \emptyset$ ; so  $W_{h-1} \subseteq D_{h-1}$ , since

$$W_{h-1} \subseteq L_{h-1} \subseteq D_{h-1} \cup J_{h-3}.$$

Since  $t$  has no neighbour in  $W_{h-1}$ , and every vertex of  $D_h$  is adjacent to every vertex of  $D_{h-1}$ , it follows that  $t \notin D_h$ . But this contradicts 12.3, since  $t$  is an  $\mathcal{S}$ -parent of  $u$ .

This proves that  $u \notin V(\mathcal{D})$ . Since  $u \in W_{h+1} \subseteq L_{h+1}$ , it follows that  $u \in J_{h-1}$ . By 12.1 applied to  $\mathcal{S}_0$ , there are two induced paths  $P_1, P_2$  of  $G$  between  $u$  and  $L_0$ , such that  $|E(P_2)| = |E(P_1)| + 2$ , and for  $j > h - 1$  every vertex in  $J_j$  that belongs to or has a neighbour in either of these paths belongs to  $D_h \cup D_{h+1} \cup \{u\}$ . Since every vertex of  $U(\mathcal{S}')$  belongs to  $L_j$  for some  $j \geq h + 2$ , except  $u$ , and  $L_j$  is disjoint from  $D_h \cup D_{h+1}$  (because  $L_j \subseteq D_j \cup (J_{j-2} \setminus V(\mathcal{D}))$ ), it follows that  $(P_1, P_2)$  is a switch for  $\mathcal{S}'$  in  $\mathcal{S}_0$ . Hence by 10.1,  $\chi(M_2) \leq \zeta$ . Since two applications of 12.2 imply that

$$\chi(M_1) \leq 2\zeta + 4((\nu + 1)(3\nu^2 + 1) + 7)\kappa,$$

it follows that

$$\chi(M') \leq 3\zeta + 6((\nu + 1)(3\nu^2 + 1) + 7)\kappa.$$

This proves 12.4. ■

**12.5** *Let  $\zeta$  be a sidekick for  $\sigma$ . Let  $\mathcal{S} = (L_0, \dots, L_k, s)$  be a stable shower in a candidate  $G$ , with mat  $M$ , such that  $\mathcal{S}$  is not  $(\sigma + 1)$ -complete over  $M$ . Let  $N = (3\nu^2 + 2)(\nu + 1) + 5$ . Then*

$$\chi(M) \leq 19201N^4\zeta + 38400N^5\kappa.$$

**Proof.** Let  $\tau = 40N\zeta + 80N^2(3\zeta + 6(N - \nu + 1)\kappa + \kappa)$ . Let  $\mathcal{W}$  be a wand in  $\mathcal{S}$ , let  $M'$  be its shadow over  $M$ , and let  $\mathcal{S}'$  be obtained by raising  $\mathcal{W}$ . Every jet of  $\mathcal{S}'$  is a jet of  $\mathcal{S}$ , and so  $\mathcal{S}'$  is not  $(\sigma + 1)$ -complete. By 12.4, the shadow over  $M'$  of every wand in  $\mathcal{S}'$  has chromatic number at most

$$3\zeta + 6(N - \nu + 1)\kappa.$$

By 11.8 applied to  $\mathcal{S}'$ , it follows that  $\chi(M') \leq \tau$ .

Thus every wand in  $\mathcal{S}$  has shadow over  $M$  with chromatic number at most  $\tau$ ; and so the result follows from 11.8, since

$$\chi(M) \leq 40N\zeta + 80N^2(\tau + \kappa) \leq 19201N^4\zeta + 38400N^5\kappa.$$

This proves 12.5. ■

Let us put these pieces together, to prove 5.1, in the following strengthened form.

**12.6** *Let  $\nu \geq 2$  and  $\kappa \geq 0$  be integers. Let  $N = (3\nu^2 + 2)(\nu + 1) + 5$ ,  $\zeta_1 = \kappa$ , and for  $1 \leq \sigma < \nu$  define*

$$\zeta_{\sigma+1} = 19201N^4\zeta_\sigma + 38400N^5\kappa.$$

*Let  $G$  be a triangle-free graph such that  $\chi^\rho(G) \leq \kappa$ , where  $\rho = 3^{\nu+2} + 4$ . If  $G$  admits no hole  $\nu$ -interval then  $\chi(G) \leq 44\nu(\kappa + \zeta_\nu)^{(\nu+1)^2} + 4\kappa$ .*

**Proof.** By 8.1,  $\zeta_1$  is a sidekick for 1. We claim that for  $1 \leq \sigma < \nu$ , if  $\zeta_\sigma$  is a sidekick for  $\sigma$  then  $\zeta_{\sigma+1}$  is a sidekick for  $\sigma + 1$ . For let  $M$  be a mat for a stable shower  $\mathcal{S}$  in a candidate  $G$ , such that  $\mathcal{S}$  is not  $(\sigma + 1)$ -complete over  $M$ . By 12.5,  $\chi(M) \leq \zeta_{\sigma+1}$ . This proves the claim that  $\zeta_{\sigma+1}$  is a sidekick for  $\sigma + 1$ . Consequently  $\zeta_\nu$  is a sidekick for  $\nu$ , and in particular, for every candidate  $G$ , every  $\nu$ -incomplete stable shower in  $G$  has floor of chromatic number at most  $\zeta_\nu$ . By 8.3, every candidate has chromatic number at most  $44\nu(\kappa + \zeta_\nu)^{(\nu+1)^2} + 4\kappa$ . This proves 12.6. ■

## 13 Acknowledgement

Thanks to Maria Chudnovsky, who worked with us on parts of the proof.

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